Optimal Inventory Control in the Presence of Dynamic Pricing and Dynamic Advertising

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Abstract

This dissertation analyzes the optimal coordination of dynamic pricing, dynamic advertising, and inventory management. We consider different optimization problems for a monopolistic retailer who faces a time-dependent deterministic demand. In Chapter 2, we generalize the model of Rajan et al. (1992). The retailer is allowed to choose a dynamic price, a dynamic advertising rate, and the inventory capacity for a sales period of fixed length so that the present value of revenue minus inventory, purchasing and (nonlinear) advertising costs is maximized; in addition, the inventory deteriorates at an exponential rate. We derive the optimal dynamic price-advertising control and the optimal inventory capacity and also consider the partially static cases when only the price is dynamic and the advertising rate is fixed and vice versa. For the optimally controlled dynamic model we carry out a sensitivity analysis with respect to the model parameters and we compare the results of the dynamic model for the optimal profit with those of the partially static models. In Chapter 3, we interpret the sales process as the controlled adoption process of a new product and the inventory capacity as untapped market share. The initial state is assumed to be exogenously given and the demand depends on the current state of the system. We exclude, however, deterioration effects and any other costs but the cost of advertising. We derive the optimal controls using a different technique than Helmes et al. (2013) - we apply Pontryagin's maximum principle. As an interesting application we consider the controlled *von Bertalanffy* model. In Chapter 4, we extend the analysis of one-period models to multi-period models and long-term average models. Assuming that the optimal controls derived in Chapter 2 and Chapter 3 are applied throughout a cycle, we treat the cycle length and the capacity as decision variables. We distinguish between the maximization of the present value of N identical cycles and the maximization of the average profit per time unit. We derive conditions that ensure the existence of an optimal pair of cycle length and capacity. Various examples and illustrations are given, and structural properties of the optimal pair are verified. For a special case we derive an extended formula of the economic order quantity.

Keywords:
dynamic pricing and advertising, inventory management, new-product adoption models, optimal control, Pontryagin's maximum principle

Zusammenfassung

Diese Dissertation analysiert das optimale Zusammenspiel dynamischer Preissetzung, dynamischer Werbung und Bestandsmanagement. Wir betrachten verschiedene Optimierungsprobleme für einen monopolistischen Händler bei gegebener zeitabhängiger deterministischer Nachfrage. In Kapitel 2 erweitern wir das Modell von Rajan et al. (1992). Der Händler darf einen dynamischen Preis, eine dynamische Werberate und die Lagergröße bei fester Verkaufsdauer wählen, so dass der Barwert von Umsatz minus Lager-, Einkaufs- und (nichtlinearen) Werbekosten maximiert wird; zusätzlich zerfällt der Lagerbestand mit exponentieller Rate. Wir ermitteln die optimale Preis-Werbe-Steuerung und die optimale Lagergröße und betrachten die semi-statischen Situationen, wenn nur der Preis dynamisch gewählt werden darf und die Werberate fix ist und umgekehrt. Wir führen eine Sensitivitätsanalyse im Hinblick auf den Einfluss der Modellparameter auf die optimalen Ergebnisse durch und vergleichen die Ergebnisse des dynamischen Modells mit denen der semi-statischen Modelle im Hinblick auf den optimalen Gewinn. In Kapitel 3 interpretieren wir den Verkaufsprozess als gesteuerten Diffusionsprozess eines neuen Produktes und die Lagergröße als unerschlossenen Marktanteil. Der Anfangszustand ist exogen gegeben und die Nachfrage hängt zusätzlich vom gegenwärtigen Zustand des Systems ab. Ein Zerfall des Lagerbestandes und alle Kosten bis auf Werbekosten sind ausgenommen. Anders als in Helmes et al. (2013) leiten wir die optimale Steuerung mithilfe des Pontrjaginschen Maximumprinzips her. Als Anwendung betrachten wir das Modell von *von Bertalanffy*. In Kapitel 4 erweitern wir die Analyse von einperiodigen Modellen auf mehrperiodige und langfristige Modelle. Die Länge des Verkaufszyklus und die Lagergröße sind Entscheidungsvariablen, wobei die optimalen Steuerungen aus Kapitel 2 und Kapitel 3 während eines Zyklus angewandt werden. Wir unterscheiden zwischen der Maximierung des Barwertes von N identischen Zyklen und der Maximierung des Durchschnittsgewinns pro Zeiteinheit. Existenzbedingungen für ein optimales Paar aus Zykluslänge und Lagergröße werden hergeleitet. Wir analysieren verschiedene Anwendungs- und Illustrationsbeispiele und verifizieren Strukturaussagen der optimalen Entscheidungsgrößen. Für einen Spezialfall leiten wir eine erweiterte Form der klassischen Losgrößenformel her.

Schlagwörter:
dynamische Preis- und Werbesetzung, Bestandsmanagement, Diffusionsmodelle, optimale Steuerung, Pontrjaginsches Maximumprinzip

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1 Introduction

Time-based pricing has always played an important role in the sales business. *Black Friday*, *season sale*, or *clearance sale* are keywords that have attracted customers for decades. In retailing these events are inextricably linked with price and quantity discounts, special buys, and *hot deals*. Customers are used to such extraordinary sales and adapt their purchasing behavior accordingly. Dynamic pricing - closely related to revenue management or yield management - is a more subtle tool than just cutting prices radically and regularly. The aim is to set prices strategically in order to maximize profits by anticipating and considering consumer behavior. In the 1980s, dynamic pricing was first applied in the airline industry which is still one of the largest fields of application. With the development and employment of computer-based pricing, storing, and booking systems, dynamic pricing found its way into many commercial domains.

The success story of the Internet has had a lasting effect on the world of retailing in the last decade. Customers who buy *on the Internet* are not reliant on opening hours, local reachability and availability, or purchase advice. On the other hand, the online retailer neither needs a physical store nor sales staff and she is able to target customers directly with promotion instruments such as newsletters or personalized advertising. Many points are taken into consideration that determine differences between classical store retailing and electronic commerce. A major feature supported by computer-based techniques is that prices can be set truly dynamically. In E-commerce, the expenditure of time and money on computer- and algorithm-based price labeling tends to zero. This does not mean that dynamic pricing comes at no cost, but the technical act of *changing* the price causes infinitesimal costs. So once an algorithm has been implemented, it is a machine's job to adjust prices. However, this flexibility in price setting is not restricted to online stores: data mining based on cash registers is standard practice in almost any shop nowadays. Many supermarkets use electronic price tags, and smart phones find a variety of application possibilities in electronic *and* store shopping. Technological progress does not only influence the pricing process in the retail industry but also affects another classical marketing instrument: advertising. Today, a company can choose from a wide range of promotion instruments, e.g., classical advertisement in a newspaper, commercials on radio and TV, and Internet-based instruments such as online advertising services or browser cookies. Furthermore, this diversity is enhanced by the opportunity to *dynamize* these advertising efforts to control consumer interest and (potential) sales in favor of the profit maximizing company. In industries such as consumer electronics or the movie business, the introduction of a new product is often accompanied by a huge promotion campaign, whereas there is only little advertising effort during the actual sales period once the product has been launched. On the other hand, local retailers often focus on promotion on special occasions or on clearing inventory and selling the remainder of stock at the end of the sales period.

The possibility of having a more extensive influence on the individual customer behavior and the aggregated sales process goes hand in hand with an increasing need to plan and manage inventories. What if a boost in sales due to an advertising campaign meets an empty inventory? Then potential sales are wasted by insufficient supply chain management. Actually, unsatisfied demand can even have a lasting negative effect on future sales since disappointed customers might not return to the store. What if a high price policy goes along with a bulging inventory? Or if only *one* product occupies most of the given shelf space? Typically, in both cases, valuable storage capacity and sales space is blocked without corresponding earnings. Furthermore, if the inventory is subject to deterioration effects, then not only does the opportunity cost lower the potential profit but additional losses are also realized because of overcapacity.

It is clear that marketing instruments must go hand in hand with production, capacity, and inventory decisions. This thesis makes a contribution to the optimal coordination of dynamic pricing, dynamic advertising, and inventory management. Throughout this thesis we consider a deterministic setting: there is no uncertainty about customer behavior, production, advertising and sales responses, or inflation. In our analysis we want to focus on the underlying characteristics and implicit decisions; that is why we exclude any stochastic disturbances. Accordingly, we are not interested in the buying decision of a particular customer but *pool* the individuals' behavior in a sales rate to represent the demand side. At this aggregated level we assume the potential buyers to be sensitive with respect to the current price p and the current advertising level w. We will also incorporate the current state x of the system, the inventory at hand, via a system function ψ to model its influence on demand. In general, we assume a demand rate λ of multiplicative form

$$\lambda(t, p, w, x) = \mu(t) p^{-\varepsilon(t)} w^{\delta(t)} \psi(x). \tag{1.1}$$

This kind of demand function is often used in the management literature. It allows the parameters of the model to be estimated by considering a log version of (1.1) and applying standard regression tools. The time-dependent function μ captures seasonal and scaling effects as well as effects that can not be influenced directly by the decision maker. The price elasticity of demand ε, $\varepsilon > 1$[1], and the nonnegative advertising elasticity of demand δ play a crucial role. They determine the market characteristics alongside the arrival intensity $\mu(t) > 0$.[2] The system function ψ acts as a *response* of the demand to the current inventory level. For example, if ψ is an increasing function, the demand is larger if the company is well stocked than if only a few items are at hand. This effect is often observed in retailing and plays an important role in shelf-space management. Incorporating the current state of the system also allows us to analyze (controlled) new-product adoption models. For such models, the main question is how the total market reacts to the introduction of a new product, how to sell goods at a profit on an individual retailer's level. In terms of the work by Selten (1965), the function ψ can be interpreted as a measure of *demand inertia*: consumers are inert with respect to price changes and also consider the (cumulative) adoption by the market. More details of the demand rate and assumptions about the parameter values and their interpretations are given in each chapter.

Throughout this thesis we consider a monopolistic retailer or producer. In Chapter 2 and Chapter 3, the sales period T is exogenously given and fixed. In Chapter 4, the sales horizon T becomes a decision variable. In Chapter 2, we are interested in finding an optimal pricing and advertising control and an inventory capacity so that the profit earned over a sales period of length T is maximized. In addition to the advertising expenditures, the monopolist has to take into account holding costs, purchasing costs, deterioration effects and discounting. In Chapter 3, we study the adoption process of new products or technologies, and the (optimal) influence of dynamic price and dynamic advertising controls on this process if the time horizon and the size of the market are fixed (and known beforehand). We consider no costs but the advertising costs and allow for general system functions ψ. For a special system function suggested by *von Bertalanffy* in the context of biological growth processes we analyze the optimal control and the system evolution in detail. In Chapter 4, we aim at maximizing the *long-term* profit by choosing a cycle length and an inventory capacity assuming that within each cycle the (optimal) controls derived in Chapter 2 and Chapter 3 are applied. We consider this optimization problem for two objective functions: the average profit per time unit as well as the present value of $N > 1$ identical cycles.

The starting point of our analysis is the model analyzed by Rajan et al. (1992) which considers joint dynamic pricing and ordering decisions. Their model is an extension of

[1] The assumption $\varepsilon > 1$ is necessary for certain statements to be derived in the following.

[2] If $p = w = \psi(x) \equiv 1$, the demand equals the function $\mu(t)$.

the model by Cohen (1977) who considers one fixed price throughout the whole sales period. Both deterministic models allow for an exponential decay of the stock and assume that the decision maker chooses the inventory capacity such that the inventory runs out exactly at the end of the sales period. Considering a single inventory cycle of fixed length $T > 0$, in Chapter 2 we generalize these models and introduce (dynamic) advertising to influence the demand rate; advertising costs are modeled by a specific nonlinear term. Moreover, we allow for time-dependent storage costs and discounting of costs and profits. We assume the system function ψ to be a constant, i.e., in Chapter 2 the sales rate does not depend on the inventory level. We derive an optimal dynamic price-advertising control that maximizes the present value of one cycle (Section 2.2) as well as optimal mixtures of dynamic and static controls, i.e., constant advertising and dynamic pricing as well as constant pricing and dynamic advertising (Section 2.3). When both marketing tools, price and advertising rate, are dynamic, the optimal price is a markup on the cost function that accounts for all the costs (except for advertising costs), for interest, and for deterioration effects. Moreover, the optimal price is not influenced by the opportunity to advertise. Under *normal* market conditions the consumers as well as the retailer benefit from advertising. More customers will purchase the product for the same price (as the optimal price is the same with or without advertising) and the retailer nets more profit by selling more items and paying only a fraction of that additional revenue as advertising expenses. Only if the consumer interest is extremely low, see Corollary 2.2.3 for details, then a retailer will sell fewer products in a market situation where advertising is effective than in a market where advertising has no effect. The optimal price, however, still remains unchanged. In particular, it is optimal for the retailer to set the prices and the advertising rates in such a way that at any time the ratio of revenue (rate) and advertising spending is proportional to the *efficiency* of advertising - measured in terms of the advertising elasticity and an advertising cost coefficient - and the price elasticity. This is a dynamic version of the well-known *Dorfman-Steiner* relation. Moreover, the net profit margin associated with an optimal control can be expressed as a product of the discount factor, the optimal advertising cost rate, and a factor depending on the *advertising efficiency*. In Section 2.4, we analyze structural properties of the optimal controls and associated values, examine the effect of parameter changes (*sensitivity analysis*), and compare the optimal dynamic price and dynamic advertising model with the optimal partially static models with the help of an illustrative example. While the major share of the additional benefit in the dynamic model stems from the opportunity to set dynamic prices, it turns out that dynamic advertising also leads to a significant increase of the total profit, especially, if the arrival intensity μ

fluctuates over time.

In Chapter 3, we drop all cost aspects except the advertising costs themselves and allow for a non-constant system function ψ, i.e., the state of the system influences the sales rate. While this framework can still be applied to inventory control problems, we introduce new-product adoption models and interpret the state of the system as the (fractional) *untapped market share*. Starting from a given initial market potential (the initial inventory or capacity), a monopolist seeks to capture the market by setting dynamic prices and dynamic advertising rates over a finite sales horizon in such a way that the present value of revenue minus advertising costs is maximized. This problem is closely related to the problem considered by Teng and Thompson (1985) who incorporate (state-dependent) learning cost effects into the production costs but assume only linear costs of advertising and do not allow for a time-dependent demand rate. Helmes et al. (2013) solve the problem by applying the Dynamic Programming method and consider the special cases when the system function $\psi(x)$ is a power function and a *Bass* functional. In this thesis, we apply *Pontryagin's maximum principle* to derive the optimal control in case of a general system function (Section 3.2). The properties of this system function determine the behavior of the optimal dynamic price path while the *cumulative* arrival intensity - the total market potential - determines the price level. It is optimal to set the advertising rate proportional to the arrival intensity $\mu(t)$. One implication of using this optimal control is that the associated path - the controlled adoption process - hits zero at the end of the planning horizon, i.e., it is optimal to act in such way that the whole market is tapped (exactly) at time T; moreover, the dynamic *Dorfman-Steiner* relation still holds for the optimal control. As an application we consider the (controlled) *von Bertalanffy* model which allows for more flexible structural properties than the *Bass* model (Section 3.3). In particular, we analyze the behavior of the optimal price function over time (*market skimming* or *market penetration*) and the question *when* prices will reach their minimum and maximum - issues that are often emphasized from a consumer's point of view. Depending on the influence of the *word-of-mouth* effect - in essence, this effect captures how strong a non-adopter is influenced by the customers who have already purchased (adopted) the product - we derive an expression that characterizes the point in time when prices peak in the time-homogeneous market situation. Before this point in time, it is optimal to strictly increase prices (*penetration strategy*) and thereafter to steadily lower prices (*skimming strategy*). If the influence of the (early) adopters is rather large, it is optimal to apply a price skimming strategy from the beginning, i.e., prices decrease over the sales horizon.

In Chapter 4, we expand our analysis from one-period models to multi-period ones and to long-term considerations. We assume that whenever the inventory is depleted at the end of one cycle, it will (instantaneously) be replenished and a new cycle begins. We introduce fixed order costs that arise once per cycle when the inventory is refilled; in particular, these costs do not depend on the volume of replenishment. While the cycle length has been arbitrary but fixed in Chapter 2 and, in addition, the capacity has also been fixed in Chapter 3, we now treat both quantities as decision variables. As far as the objective is concerned, we distinguish between the maximization of the present value of N identical cycles, N finite or infinite, and the maximization of the average profit per time unit.

In Section 4.2, we assume that throughout each cycle of length τ - the change in notation indicates that the cycle length is now a decision variable in contrast to the exogenously given value T - the profit maximizing dynamic price and advertising controls derived in Chapter 2 are applied. The expressions to be examined depend on the optimal net profit margin associated with these optimal controls rather than the controls themselves. Since the value of the (optimal) capacity is implicitly defined by the (optimal) controls and the choice of the cycle length, see above, we face a (nonlinear) one-dimensional optimization problem. We derive necessary and sufficient conditions that guarantee the existence of a (unique) optimal cycle length (Theorems 4.2.1, 4.2.2, and 4.2.3). Although we base our analysis on the optimal control derived in Chapter 2, it is important to note that for *any* profit rate satisfying the optimality conditions a (sub)optimal cycle length is characterized. Thus, our framework is not restricted to the particular case when the controls throughout the cycle are optimal but applies to a broader class of inventory control systems. We illustrate our theoretical findings with examples that shall emphasize the influence of the arrival intensity and we derive structural properties of the optimal cycle length value for special parameter settings. In particular, we show that if the optimal advertising rate lies above a threshold determined by the advertising efficiency, then the opportunity to advertise benefits the retailer also in the long run and leads to shorter inventory cycles. Due to shorter inventory cycles, customers also benefit because of fresher products while prices remain unaffected (Theorem 4.2.4).

In Section 4.3, we assume that the revenue maximizing controls derived in Chapter 3 are applied throughout each inventory cycle. Since the revenue maximizing controls account for no costs except for the costs of advertising, we evaluate the running costs along the optimal path; the purchasing costs and the order costs have to be paid at the beginning of each cycle. While the order costs are fixed and the costs of purchasing (or production) only depend on the inventory volume, the running costs depend on the capacity *and* the cycle length. Also the revenue of each cycle - the expression for the value function at time zero derived in Chapter 3 - depends on both quantities, the capacity and the cycle length. Similarly to Section 4.2, we consider the N cycle problem and the time-average problem. However, the resulting optimization problems are now two-dimensional; the cycle length and the capacity are decision variables. We first consider one-dimensional subproblems by keeping one of the decision variables fixed. We derive optimality conditions that ensure the existence of an optimal cycle length if the inventory capacity is fixed. Alternatively, we keep the cycle length fixed and formulate optimality conditions for a capacity to be optimal. Finally, we formulate optimality conditions for the two-dimensional case. These conditions are quite general. We illustrate our results with examples. In a particular parameter setting, we are able to derive explicit solution formulas for the optimal cycle length value and the optimal capacity value. We refer to this result as the *endogenized Harris-Wilson* formula (Proposition 4.3.8).

When considering multi-period decisions, we pay particular attention to the question whether or not an order scheme is *feasible*, i.e., if a pair of cycle length and capacity values guarantees nonnegative profits. Since the (optimal) profit rate derived in Chapter 2 is strictly positive and takes all costs, except the (fixed) setup costs, into account, in Section 4.2 an order scheme is feasible if the cumulative profits over one cycle cover these setup costs. The analysis in Section 4.3 is based on the revenue maximizing policies derived in Chapter 3 which do not consider any costs (except the costs of advertising). Thus, inventory costs have to be evaluated along the path associated with a particular cycle length and capacity value. Hence, it becomes more difficult to determine whether or not an order scheme is feasible. The order schemes derived in Section 4.3 are only suboptimal compared to those which solve the profit maximizing problem. However, the former allow for a state-dependent demand rate, whereas in Chapter 2 we assume $\psi(x) \equiv 1$, see above. Moreover, these suboptimal solutions will lead to a reasonable decision rule in many applied cases.

A literature review focusing on corresponding topics can be found in the introduction of each chapter; particular references are also given in each section. We typically consider a monopolistic retailer who faces joint pricing, advertising, and inventory decisions. In Chapter 3, we focus on new-product adoption models by interpreting the inventory as an untapped (fractional) market share. However, this thesis makes a contribution to many economic applications and situations: a producer who has to optimally set a production capacity and faces marketing decisions, a deterministic news-vendor who may choose the order quantity and the (dynamic) price and advertising rate, a manager who faces a resource extraction problem, or a decision maker who aims for the optimal forest rotation age. By abstracting from price and advertising as specific interpretations of the controls, our models and solutions apply to different subjects. Here is another example: tool wear problems, where (1.1) describes the wear-out effect and the goal is to control the wearing process so that a (utility) function is maximized.

A precise description of the *monopolistic retailer framework* together with explanations and interpretations of the model's parameters and assumptions is given in Chapter 2.

2 Optimal Dynamic Pricing and Advertising with Inventory Cost

2.1 Introduction

As a starting point of our analysis, we introduce the model by Rajan et al. (1992). These authors consider a monopolistic retailer who is allowed to choose the length of an inventory cycle, the initial inventory level, and a dynamic price control $p(t)$ throughout the cycle, cf. Chapter 1. The time- and price-dependent deterministic demand rate $\lambda_R(p,t)$ is assumed to be a nonincreasing separable function of price and time.[1] A key assumption which underlies the analysis by Rajan et al. (1992) is that the value of the initial inventory at the beginning of a cycle is solely subject to the decision of the monopolist. This assumption and the fact that the demand rate does not depend on the inventory level allows Rajan and co-authors to solve their problem in two steps. First, the cycle length $T > 0$, T finite, is assumed to be fixed and the objective is to choose a dynamic price control that maximizes

$$\int_0^T (p(t) - c_R(t))\,\lambda_R(p(t),t)dt, \tag{2.1}$$

where $c_R(t)$ is a cost functional; the function $c_R(t)$ takes constant unit costs and inventory costs affected by deterioration effects into account. The precise definition and derivation of the function $c_R(t)$ is given below. The inventory level $x(t)$ at time $t \in [0,T]$ satisfies the following differential equation with terminal condition $x(T) = 0$:

$$\dot{x}(t) = -\lambda_R(p(t),t) - q(t)x(t), \qquad x(T) = 0,\, x(0) > 0; \tag{2.2}$$

the *wastage coefficient* $q(t) \geqslant 0$ may vary over time. Thus, the initial inventory associated with a pricing policy $p(t)$ and duration T is implicitly given by the solution of (2.2). The

[1] Rajan et al. (1992) require the demand rate not to increase in t to guarantee the uniqueness of the optimal cycle length, see below. For the solution of their dynamic pricing problem it is sufficient to assume that λ_R is nonincreasing in p only.

optimal pricing strategy which maximizes (2.1) yields the associated value of the optimal initial inventory. In the second step, Rajan et al. (1992) maximize the expression

$$\frac{1}{\tau}\left[\int_0^\tau \left(p(t)-c_R(t)\right)\lambda_R(p(t),t)dt-k\right] \tag{2.3}$$

with respect to the cycle length τ applying the optimal price policy obtained in step one, cf. Chapter 4. The setup cost $k>0$ has to be paid once per cycle.

We will extend their model by assuming the demand to depend explicitly on an advertising component. Thus, the monopolist must set the advertising level $w(t)\geqslant 0, t\geqslant 0$, and incurs an advertising cost (rate) of $w(t)^{a(t)}, a(t)>0$. The function $a(t)$ is assumed to be piecewise continuous, strictly positive, and bounded from above, see below. In contrast to the situation where no active promotion is considered - which can be thought of as an external advertising factor of one - our model assumes that the monopolist can boost the sales rate by paying for the promotional effort made. Actually, the monopolist is obliged to promote her product as otherwise demand will decline or even dry up. So it is not clear under which circumstances the decision maker will benefit from the opportunity to advertise and whether she will benefit at all. Therefore, it is difficult to compare results of our model with those of the pure pricing model. Nevertheless, we will compare both models by assuming certain demand structures.

Our analysis will be split into two parts. In the following Section 2.2, we first consider the dynamic pricing and advertising problem with a given time horizon $[0,T]$. We will examine the case of a multiplicative demand rate extending the pricing problem considered by Rajan et al. (1992) by an advertising component. In addition, we incorporate time-dependent elasticities and time-dependent inventory cost. Throughout this thesis all functions will be assumed to be piecewise continuous.[2]

All revenues and costs are subject to a time-dependent discount rate $r(t)\geqslant 0$.[3] We define the cumulative discount rate $R(t):=\int_0^t r(s)ds$ and a future payment t periods ahead is discounted by the factor $\xi(t):=e^{-R(t)}$. Setting $r(t)\equiv r$ results in the *common* exponential discounting, $\xi(t)=e^{-rt}$. The time-dependence of the discount rate enables us to consider also other discounting methods, e.g., hyperbolic discounting. Hyperbolic discounting models time inconsistencies in the behavior of economic agents, i.e., the decision between two alternative payments will not only depend on the size of the payment

[2] We call a function piecewise continuous if it has finite values and at most a finite number of discontinuities on its domain and if the function is bounded on any compact interval.

[3] Since Rajan et al. (1992) aim for the (time) average problem they abstain from considering any discounting effects explicitly, i.e., $r(t)\equiv 0$. Nevertheless, their *value drop* component, see below, can be interpreted as acting like a discounting factor.

but also on the point in time when the payment is made. A common observation is that people become more *patient* when choosing between a smaller reward earlier and a larger reward later if the delay is the same but occurs later: when choosing between one dollar today and two dollars tomorrow a certain fraction of all agents prefers the one dollar today. But having the choice between one dollar in one year or two dollars in one year and one day, most agents will prefer the two dollars; their decision is time inconsistent. Frederick et al. (2002) give a review of different approaches to discounting and time preferences and discuss several functional forms ξ_h of hyperbolic discounting that have been proposed in the literature. For example, Loewenstein and Prelec (1992) suggest $\xi_h(t) := 1/\left(1+\alpha t\right)^{\beta/\alpha}$, $\alpha, \beta > 0$, where α determines the extent to which the hyperbolic discount deviates from the *common* exponential discount function[4]; note, $\lim_{\alpha \to 0} \xi_h(t) = e^{-\beta t}$. Then, today's present value of a payment of size one paid tomorrow $(t = 1)$ equals $\xi_h(1)/\xi_h(0) = \left(\frac{1}{1+\alpha}\right)^{\beta/\alpha}$. The present value at time $t > 1$ of a payment of size one paid at $t+1$ is given by $\xi_h(t+1)/\xi_h(t) = \left(\frac{1+\alpha t}{1+\alpha(t+1)}\right)^{\beta/\alpha} = \left(1 - \frac{\alpha}{1+\alpha(t+1)}\right)^{\beta/\alpha}$, which can easily be proved to be larger than $\xi_h(1)/\xi_h(0)$, i.e., the agent becomes more patient when time increases. For the classical concept of exponential discounting with $r(t) \equiv r$ - in contrast to hyperbolic discounting - the valuation is independent of the point in time since $\xi(1)/\xi(0) = \xi(t+1)/\xi(t) = e^{-r}, t > 0$. In the following, we will assume the discounting rate to be of the form $e^{-R(t)}$, $R(t)$ defined as above. To model the situation of hyperbolic discounting one can simply choose $r(t) \equiv \frac{\beta}{1+\alpha t}, \alpha, \beta > 0, t \geqslant 0$. Then, $R(t) = \frac{\beta}{\alpha}\log(1+\alpha t)$ and $e^{-R(t)} = \xi_h(t)$.

While Rajan et al. (1992) assume the demand rate λ_R to be a nonincreasing function of time, see above, we also allow for demand rates that are increasing in time, i.e., assuming a constant price and a constant advertising rate, an increasing number of customers is willing to purchase the product for the same price and marketing level. For the general dynamic pricing and dynamic advertising problem and for special cases and parameter settings, we derive sensitivity results and study numerical examples. We will also address the problem of determining an optimal *static* advertising rate, i.e., a constant advertising rate throughout the whole cycle, while prices are still dynamic. This enables us to compare the two models - the pure pricing model analyzed by Rajan et al. (1992) and our (general) model. We complete the analysis of the one-cycle problem by also considering static prices and dynamic advertising rates.

In Chapter 4, we face the inventory control problem of determining a cycle length that maximizes the average profit per unit of time when the optimal price and advertising

[4]Other possible choices are, for example, $\xi_h(t) = 1/t$, or $\xi_h(t) = 1/(1+\alpha t), \alpha > 0$, see Frederick et al. (2002), pp. 360.

schemes are applied. We will give conditions under which the existence and uniqueness of an (optimal) finite cycle length can be guaranteed.

2.2 The Dynamic Pricing and Advertising Model

We extend the pricing problem in Rajan et al. (1992), cf. Section 2.1, by the opportunity to advertise. A monopolist wants to maximize her profit by choosing a pricing and advertising scheme and the initial inventory level under the consideration of production and inventory costs. In addition, she has to pay for the promotion which she is able to control. The length of the finite time horizon is denoted by T and is assumed to be fixed. We assume a time-dependent discount rate $r(t) \geqslant 0$ and we let $R(t) = \int_0^t r(s)ds$ denote the cumulative discount rate, see above.

The deterministic demand rate $\lambda \geqslant 0$ is of multiplicative form and depends on the control $u(t) = (p(t), w(t))$, the price p and the advertising effort w at time t, $0 \leqslant t \leqslant T$, and time itself, i.e.,

$$\lambda^{(u)}(t) := \lambda(t, u(t)) = \lambda(t, p(t), w(t)) = \mu(t)p(t)^{-\varepsilon(t)}w(t)^{\delta(t)}, \tag{2.4}$$

where $\infty > \varepsilon(t) \geqslant \underline{\varepsilon} > 1$ is the (time-dependent) price elasticity and $0 \leqslant \delta(t) < a(t) \leqslant \overline{a}$ is the advertising elasticity; $a(t)$ is a cost parameter to be specified below. The parameter functions $a(t), \delta(t)$, and $\varepsilon(t)$ are assumed to be piecewise continuous, see above. The lower bound $\underline{\varepsilon}$ and the upper bound $\overline{a}$ are assumed to be arbitrary but fixed numbers. These bounds ensure the integrability of expressions wherever $a(t), \delta(t)$, and $\varepsilon(t)$ show up in an exponent. The demand (2.4) is a special case of (1.1) when $\psi(x) \equiv 1$, i.e., the state of the system - the inventory level - has no influence on the demand. Let $\Delta(t) := \frac{\delta(t)}{a(t)}$; this (parameter) function quantifies the *advertising efficiency*. Note, the function $\Delta(t)$ only takes values in the interval $[0, 1)$. An increasing price elasticity captures situations where customers become more price sensitive over time. This might be the case with *fashion* goods: at the beginning of the season the willingness to pay is high as the fashion is new and trendy. As time passes, more and more people adopt the new fashion and it becomes less desirable to buy the dated goods. On the other hand, when selling airline tickets the customer's price elasticity typically decreases over time. At the beginning of the sales horizon the (potential) passengers are flexible and they might choose another airline or flight if the price is too high. The nearer to the departure date the less flexible the customers are, so that - especially - business travelers are willing to pay very high prices if they need a flight the next day. In both cases, the price elasticity may not only depend on current time but also on the length of the sales period, i.e., $\varepsilon(t) := \varepsilon(t, T)$.[5] Although we do not explicitly consider adoption effects - the demand does not depend on the inventory level or the amount already sold since $\psi(x) \equiv 1$, cf. (3.4) - such kind of effects can (somewhat) be modeled by a properly chosen time-dependent elasticity as indicated in the fashion example above. Likewise, the time dependence of the advertising elasticity accounts for the fact that customer's response to advertising may change over time. For instance, to go in line again with the fashion example, promoting a brand new product might yield a larger effect than promoting an already established product. By allowing the cost coefficient $a(t)$ to depend on time one is able, for example, to model the diminishing advertising effect over time in case of a (nearly) constant δ. It becomes more expensive over time to maintain a given advertising rate. The specific values of $a(t)$ and $\delta(t)$ matter only in particular cases; typically, the ratio of the two functions, the advertising efficiency $\Delta(t)$, is important.

Actually, we could parametrize our model simply in terms of the advertising efficiency Δ. The cost term to be considered in the objective (2.6), see below, is $w(t)^{a(t)}$; the amount of dollars that has to be paid to advertise at a rate $w(t)$ at time t.[6] By introducing $W(t) := w(t)^{a(t)}$ as the decision variable the factor $w(t)^{\delta(t)}$ in the demand rate can be replaced by the factor $W(t)^{\Delta(t)}$. Then, the number of parameters has been reduced by one since only the efficiency parameter Δ has to be considered. However, we retain the two-dimensional parametrization in a and δ to be able to compare many specific models. For instance, Sethi et al. (2008) consider a quadratic cost function and a linear factor for the advertising rate ($a(t) \equiv 2$ and $\delta(t) \equiv 1$) while Vidale and Wolfe (1957) emphasize the linear cost structure ($a(t) \equiv 1$), cf. Appendix 2. In particular, the two parameters a and δ enable us to distinguish between the effect of advertising (measured in terms of the exponent δ) and the spending on advertising (measured in terms of the exponent a).

We call $\gamma(t) := \frac{\varepsilon(t)-\Delta(t)}{1-\Delta(t)}$ the *leveraged price elasticity of demand*. This elasticity describes the effect of the interaction of both marketing instruments, price and advertising. If $\delta = \Delta = 0$, the leveraged price elasticity of demand simply equals the price elasticity of demand - a natural result since advertising has no effect and the demand rate is only time- and price-dependent. Lacking any impact, it is optimal to set the advertising rate zero equal to zero. Then, the model becomes the pure pricing model considered in Rajan et al. (1992). If Δ is positive, advertising has an impact; this impact is quantified by the value of γ. Note, γ increases in ε and Δ.

[5] In the following, we will only write $\varepsilon(t, T)$ if we want to put the focus on the cycle length dependence. Apart from that, we adhere to write $\varepsilon(t)$ to not overload the notation.

[6] Note, that $w(t)^{a(t)}$ can be interpreted as thousands or millions of dollars.

The time-dependent expression $\mu(t) > 0$ reflects a possibly seasonal component or a structural influence on demand that is not subject to the decision by the monopolist; different effects can hereby interfere. In specific applications, μ may have a product form where each factor has its own economic interpretation. For instance, if $\mu(t) = \mu_1\mu_2\mu_3(t)$ the constant μ_2 can represent the average flow of customers; the time-dependent μ_3 can be thought of as the seasonal component and μ_1 is the so-called *response constant*, cf. Chapter 3. The classical interpretation of this response constant is the number of sales per advertising dollar spent. This interpretation of the first factor μ_1 carries over to our case. In the Vidale-Wolfe model, cf. Appendix 2, w linearly enters the dynamics and the objective, while in our case w enters the dynamics and the objective function as a power expression with exponent $\delta(t)$ and $a(t)$, respectively. Equation (3.67) in Chapter 3 provides another example of a μ function which is motivated by the model of Bemmaor (1994).

We call μ the *arrival intensity* or the *basic demand*. It reflects the standard buying behavior of customers over time. All the above-mentioned factors are *pooled* in $\mu(t)$ and enter the demand rate (2.4). Furthermore, this innocent looking function μ provides the opportunity to model diverse effects. Given an advertising rate of value one and a price of value one, e.g., a break even price relative to costs, see below, the demand rate equals $\mu(t)$. Due to the fact that the price enters the demand as a power function expression, namely $p^{-\varepsilon(t)}$, this expression can become very small for large price values and/or large values of ε. For example, (grocery) discounter stores face customers that are very price sensitive: if the prices for convenience goods, which are usually store brands, increase (too much), the customers simply change their local supplier; an (interstore or intrabrand) price elasticity near ten is not uncommon. For example, Bucklin and Srinivasan (1991), based on survey data, estimate the price elasticity for various coffee brands to range from minus twenty to minus five.[7] As a consequence, the values of the μ function can be very large, and the numerical analysis becomes challenging: considering a (large) price p, no advertising influence ($\delta \equiv 0$) and a value of $\varepsilon = 10$, the function μ must take a value of p to the power 10 in order to represent a total demand rate of one unit. Therefore, p can be interpreted as a price *relative* to some reference price level p_0 or the unit cost c_0, see below. Then, $p = \hat{p}/p_0$ or $p = \hat{p}/c_0$, and $\hat{p}$ is the price observed by customers. If p is defined as such a fraction, p will take values close to one and the power expression with large exponents is much easier to handle. A side benefit of this relative price interpretation is that μ can be interpreted as a natural or market-driven

[7] Taken the whole market, the (long-run) demand of convenience goods is assumed to react (particularly) inelastic to price changes with values above -1, see, for example, Table 5.3 in Mansfield (1994).

sales potential, i.e., the rate of customers willing to buy the product at unit price and unit advertising rate.

An example for declining demand over time is the case of selling certain perishable products, e.g., fruits or flowers (or fashion).[8] At the beginning of the sales period the goods are fresh and attractive, and demand is relatively high; over time, the goods perish or become outdated, and fewer customers are willing to spend their money (in addition to the effect that customers become possibly more price sensitive as indicated above). Rajan et al. (1992) model such situations by introducing an increasing function μ_r which they call the *value drop* component. Rajan et al. postulate that the price to be observed by the customers is given by $p_r(t,p) := \mu_r(t) * p$; p is the price set by the retailer. Since p_r increases over time the rate of sales associated with a fixed price p_r decreases over time (the value of the price p drops over time). Except for the value drop component and the price to be chosen, Rajan et al. (1992) allow no other time dependence on the demand. In contrast, we allow explicitly for a time component $\mu(t)$, which might also increase over time. We assume $\mu(t)$ to be a positive piecewise continuous function on the interval $[0,T]$. Naturally, the demand decreases in p and increases in w; the latter only if $\delta(t) > 0$. Recall, throughout Chapter 2, we do not assume that the demand rate depends explicitly on the current value of the inventory, i.e., we consider a special case of the demand function considered in Chapter 3; we set $\psi(x) \equiv 1$. Let $c_0 > 0$ denote the cost of production (or purchasing) per unit, and let $\ell(t) \geqslant 0$ be the (time-dependent) carrying cost rate per unit. This function $\ell(t)$ not only captures the costs of physically storing the goods but might also include the cost of capital, e.g., the interest on working capital, (inventory level dependent) insurance costs, costs for labeling prices, etc. We assume that the advertising costs accrue at a rate $w(t)^{a(t)}$, which allows for convex and concave cost functions. In practice, the monopolist can potentially spend her money on a large number of advertising efforts. For example, these expenditures might be the cash flow rate an advertising agency receives in order to run a campaign. Alternatively, the budget flow can be spend on an internal advertising department. Another interpretation could be the number of assistants in a shop advising (potential) buyers, or the length of the assistants' working time. The longer assistants are working the larger the operating costs - especially wages - which (to a certain degree) can be assigned to the advertising costs. If $\delta(t) > 0$, a retailer's decision to spend no money on advertising is then interpreted as closing the shop and to run no business at all.

We assume the demand to be deterministic. However, one interpretation of the demand (2.4) is that the advertising efforts influence the number of people who are willing

[8] Perishable products are products that worsen in quality over time, and become lesser in value.

to buy the product, and the price level influences the probability of purchasing.

The restriction $a(t) > \delta(t)$, $t \in [0,T]$, on the parameter functions ensures that the *leverage effect* of advertising remains bounded; otherwise, there is the incentive to increase the promotional spending to infinity in order to increase demand, and thus the revenue would also tend towards infinity. Let $x(t), 0 \leqslant t \leqslant T$, denote the inventory level at time t. This is the state of the system. We do not allow for backlogging, i.e., $x(t) \geqslant 0$. The initial inventory level is denoted by x_0, i.e., $x(0) = x_0$, and is subject to the decision of the monopolist. In order to explicitly account for the perishability of the items to be sold we model the inventory level at time t to deteriorate at a rate $q(t) \geqslant 0$. We assume $q(t)$ as well as $\ell(t)$ and $r(t)$ to be nonnegative piecewise continuous functions and thus to be integrable on $[0,T]$. For any admissible control u, see below, the inventory process evolves as

$$\dot{x}(t) = -\lambda^{(u)}(t) - q(t)x(t), \qquad x(T) = 0, \tag{2.5}$$

where $\dot{x}(t) = dx(t)/dt$ denotes the derivative with respect to t. Note, the terminal condition together with $\lambda^{(u)}(t) \geqslant 0$ and $q(t) \geqslant 0$ implies the inventory process to be nonnegative, i.e., no backordering will occur. For T arbitrary but fixed, let U_T denote the set of feasible controls:

$$U_T = \left\{ u \;\middle|\; \begin{array}{l} u(t) = (p(t), w(t)), \text{ and } u(t) \text{ is a vector-valued piecewise continuous} \\ \text{function on } [0,T], 0 \leqslant t \leqslant T, \text{ such that the solution of (2.5) with} \\ \text{respect to } x(t) \text{ is uniquely determined and all integrals to be} \\ \text{encountered in the sequel exist; moreover, } p(t) > 0, w(t) \geqslant 0 \end{array} \right\}.$$ [9]

The decision problem is to choose a control $u(t) = (p(t), w(t)) \in U_T, 0 \leqslant t \leqslant T$, that maximizes

$$\pi_1(T,u) = \int_0^T e^{-R(t)} \left[p(t)\lambda^{(u)}(t) - \ell(t)x(t) - w(t)^{a(t)} \right] dt - c_0 x_0. \tag{2.6}$$

The costs of purchase $c_0 x_0$ have to be paid at the beginning of the sales period and are thus not discounted.[10] Although this problem looks very similar to the control

[9] While we are interested in determining optimal price and advertising policies we could as well consider the demand and advertising rate as controls. Then, the price associated with a demand-advertising policy is obtained by rewriting (2.4). This corresponds to a quantity based approach often applied in microeconomics since optimizing over the demand means essentially choosing an initial inventory level (or capacity). Influenced by the literature on revenue management we choose price and advertising as controls and deduce the demand via (2.4), but both approaches are possible (and lead to the same results).

[10] Recall, instead of costs of purchase one can think of production costs.

problem which we are going to consider in Chapter 3 there is a crucial difference: the initial inventory level (or capacity) x_0 is not exogenously given but will be implicitly determined by the price and advertising control the monopolist chooses. Equation (2.7), see below, defines the relation between an admissible control u and the associated state process. To see why it is reasonable to assume the terminal condition $x(T) = 0$, cf. (2.5), consider the monopolist who chooses $x(0) = x_A$ together with a control u_A such that $x(T) =: x_{T_A} > 0$. Then, the same pricing and advertising scheme can be run but with initial inventory $x(0) = x_B = x_A - x_{T_A}$. The revenue is the same for both initial values as the demand rate remains unchanged (it is independent of the inventory level), but the costs will decrease: the level of the inventory process initiated with $x(0) = x_B$ is below the one starting at $x(0) = x_A$ for every point in time, i.e., the production and inventory costs are smaller. However, as the clearance rate of the inventory does also depend on the current level through $q(t) \geqslant 0$, this will not imply that $x(T) = 0$ (but guarantees that $x(T) \geqslant 0$). Thus, it is optimal to control the inventory in such way, that it is depleted at the end of the sales horizon, i.e., $x(T) = 0$. Making use of the state equation (2.5) and the terminal condition we are able to rewrite the objective function (2.6) solely in terms of the controls (and parameters) and independent of the current state of the system.

Let $Q(t) := \int_0^t q(s)ds$ denote the cumulative loss rate. Then, $0 \leqslant t \leqslant T$,

$$\frac{d}{dt}\left(x(t)e^{Q(t)}\right) = e^{Q(t)}\dot{x}(t) + q(t)x(t)e^{Q(t)} = -e^{Q(t)}\left(-\dot{x}(t) - q(t)x(t)\right) = -e^{Q(t)}\lambda^{(u)}(t),$$

where me make use of (2.5). Integrating from t to T and using the boundary condition $x\,(T) = 0$ we obtain

$$-x(t)e^{Q(t)} = \int_t^T -e^{Q(s)}\lambda^{(u)}(s)ds.$$

Thus, the controlled inventory process follows the trajectory

$$x(t) = e^{-Q(t)}\int_t^T e^{Q(s)}\lambda^{(u)}(s)ds. \tag{2.7}$$

Equation (2.7) is important as it links the inventory level at time t with the values of the pricing and advertising policies on $[t, T]$. Formula (2.7) determines the initial inventory level (or capacity) x_0 for any given control u, especially for the optimal control. Now, we are able to quantify the inventory cost of the trajectory $x(t)$, see (2.6), in terms of the demand rate by applying Fubini's theorem:

$$\begin{aligned}\int_0^T e^{-R(t)}\ell(t)x(t)dt &= \int_0^T e^{-R(t)}\ell(t)\left(e^{-Q(t)}\int_t^T e^{Q(s)}\lambda^{(u)}(s)ds\right)dt\\ &= \int_0^T\int_t^T e^{-R(t)}\ell(t)e^{-Q(t)}e^{Q(s)}\lambda^{(u)}(s)dsdt\\ &= \int_0^T\int_0^s e^{-R(t)}\ell(t)e^{-Q(t)}e^{Q(s)}\lambda^{(u)}(s)dtds\\ &= \int_0^T \lambda^{(u)}(s)\left(e^{Q(s)}\int_0^s e^{-(Q(t)+R(t))}\ell(t)dt\right)ds.\end{aligned}$$

We introduce the cost function

$$c(t) := e^{Q(t)+R(t)}\left(c_0+\int_0^t e^{-(Q(s)+R(s))}\ell(s)ds\right). \tag{2.8}$$

Expression (2.8) can be rewritten as $e^{Q(t)}\int_0^t e^{-(Q(s)+R(s))}\ell(s)ds = e^{-R(t)}c(t)-c_0e^{Q(t)}$ so that the inventory cost of the trajectory $x(t)$, see above, can be expressed as

$$\int_0^T e^{-R(t)}\ell(t)x(t)dt = \int_0^T e^{-R(t)}c(t)\lambda^{(u)}(t)dt - c_0\int_0^T e^{Q(t)}\lambda^{(u)}(t)dt.$$

Eventually, since $x_0 = x(0) = \int_0^T e^{Q(t)}\lambda^{(u)}(t)dt$, we can rewrite the objective (2.6) as follows:

$$\begin{aligned}\pi_1(T,u) &= \int_0^T e^{-R(t)}\left[p(t)\lambda^{(u)}(t)-\ell(t)x(t)-w(t)^{a(t)}\right]dt - c_0x_0\\ &= \int_0^T e^{-R(t)}\left[p(t)\lambda^{(u)}(t)-\ell(t)x(t)-w(t)^{a(t)}\right]dt - c_0\int_0^T e^{Q(t)}\lambda^{(u)}(t)\,dt\\ &= \int_0^T e^{-R(t)}\left[(p(t)-c(t))\,\lambda^{(u)}(t)-w(t)^{a(t)}\right]dt.\end{aligned}$$

The cost functional $c(t)$ is independent of the controls and captures all inventory-dependent costs: $c(t)$ is the future value of the costs related to the sale of one unit at time t. It is the sum of the future value of the unit cost and the future value of the present value of the inventory cost at time t. The costs associated with the deterioration of the stock are taken into account by the factor $e^{Q(t)}$: to sell one unit at time t the monopolist must store $e^{Q(t)}$ units at the beginning. For example, if the inventory decays at a rate of 10%, the monopolist must stock exp(1) units at the beginning in order to be able to sell one unit at $t = 10$, i.e., more than 60% *vanish* and inflict nothing but costs.

The cost function is the equivalent of the function $c_R(t)$ in (2.1) where the inventory cost is constant and no discounting is considered, i.e., $\ell(t) \equiv \ell$ and $r(t) = R(t) \equiv 0$. Expression (2.8) reveals that the terms $Q(t)$ and $R(t)$, the cumulative deterioration rate and the cumulative interest rate, act in a similar way: the inventory decay can be thought of as an additional interest factor - or the interest rate acts like an additional decay rate. So for the evaluation of the purchase and inventory costs it does not matter whether the discount rate or the decay rate is larger, but what matters is the sum of both quantities. However, from a practitioner's point of view it certainly is of interest if higher costs arise from a higher inventory level (as a consequence of a larger decay rate) or are simply due to discounting.

The function $c(t)$ is differentiable[11] and increasing in t since

$$\dot{c}(t) = (q(t) + r(t))\, c(t) + \ell(t) \geqslant 0, \tag{2.9}$$

$c(0) = c_0$; $c(T) < \infty$ if T is finite, and $\lim\limits_{t\to\infty} c(t) = +\infty$ if $\ell(t) + q(t) + r(t) > 0$. In the time-homogeneous case, i.e., $\ell(t) \equiv \ell, q(t) \equiv q$, and $r(t) \equiv r$, $c(t)$ simplifies to

$$c(t) = \begin{cases} e^{(q+r)t}\left[c_0 + \frac{\ell}{q+r}\left(1 - e^{-(q+r)t}\right)\right], & \text{if } \ell \geqslant 0, q + r > 0, \\ c_0 + \ell t, & \text{if } \ell \geqslant 0, q = r = 0. \end{cases}$$

If $\ell = q = r = 0$, the cost function $c(t)$ is constant and it only accounts for the unit cost c_0; if $\ell > 0$, it increases linearly over time at rate ℓ. The cost function together with the function μ is of great importance as the (optimal) controls will depend on $c(t)$. Thus, the cost structure and associated parameters will influence the behavior of the system over time. Figure 2.1 depicts the cost function $c(t)$ on the interval $[0, 10]$ for different parameter settings; the unit cost equals one, i.e., $c_0 \equiv 1$.

We define the instantaneous *profit margin* ν; in the pure pricing model it was introduced by Rajan et al. (1992). Now, it also accounts for the (discounted) advertising

[11] Except at most finitely many points.

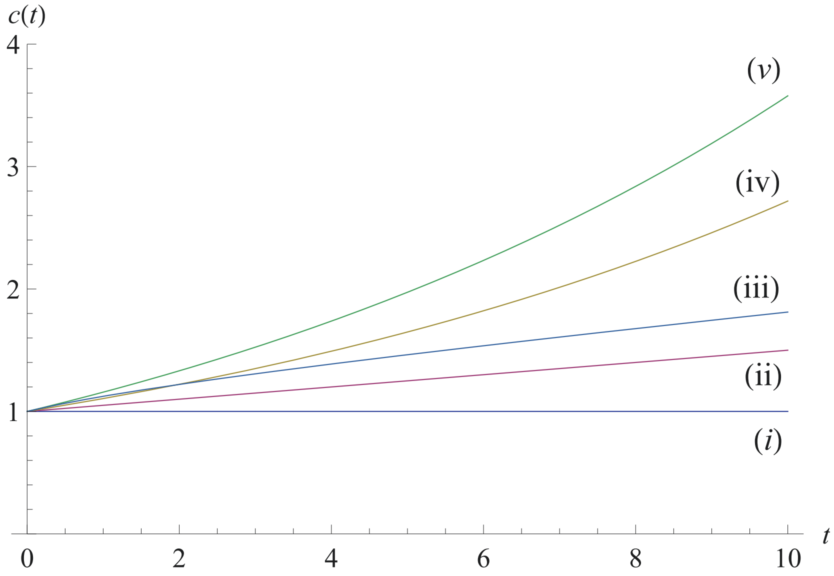


Figure 2.1: The cost function $c(t)$, see (2.8), where $c_0 \equiv 1$ and (i) $\ell(t) = q(t) = r(t) = 0$, (ii) $\ell(t) = 0.05, q(t) = r(t) = 0$, (iii) $\ell(t) = 0.05, q(t) = 0, r(t) = \frac{0.1}{1+t}$, (iv) $\ell(t) = 0, q(t) + r(t) = 0.1$, (v) $\ell(t) = 0.05, q(t) + r(t) = 0.1$.

spending:

$$\nu(t,u) = \nu(t,p,w) := e^{-R(t)} \left[(p - c(t))\, \lambda\,(t,p,w) - w^{a(t)} \right]. \tag{2.10}$$

The profit margin ν is piecewise continuous in t and differentiable in the arguments p and w. Let $\pi_1^*(T) := \pi_1(T, u^*)$ denote the optimal value of the control problem (2.6), and let $u^* = (p^*, w^*)$ denote the optimal control[12], i.e.,

$$\pi_1^*(T) = \sup_{u \in U_T} \int_0^T e^{-R(t)} \left[(p(t) - c(t))\, \lambda^{(u)}(t) - w(t)^{a(t)} \right] dt \tag{2.11}$$

$$= \sup_{u \in U_T} \int_0^T \nu(t,u) dt$$

$$\leqslant \int_0^T \sup_{\{p>0, w \geqslant 0\}} \{\nu(t,p,w)\}\, dt. \tag{2.12}$$

Although T is exogenously given, we stress the dependence of the (optimal) profit on

[12]The existence and finiteness of the optimal value and the existence and uniqueness of the optimal control are proved below.

the sales period T. Later on, we are interested in choosing an optimal cycle length that maximizes, for example, the average profit per time unit, cf. Chapter 4. Given the upper bound (2.12) on $\pi_1^*(T)$, we consider for every point in time t the auxiliary problem of finding pointwise solutions $(p^*(t), w^*(t))$ which satisfy

$$(p^*(t), w^*(t)) = \underset{p>0, w\geqslant 0}{\arg\max}\, \nu(t, p, w). \tag{2.13}$$

In the following, we will show that under (relatively mild) parameter restrictions the control u^*, where $u^*(t) = (p^*(t), w^*(t))$ satisfies (2.13) pointwise, is a feasible policy in U_T, and thus solves problem (2.11). Let $\lambda^*(t) := \lambda(t, u^*(t))$ denote the sales rate and let $\nu^*(t) := \nu(t, u^*(t))$ be the profit margin associated with the (pointwise) optimal control at time t. From the first order conditions for p^* and w^* being optimal, see below, the dynamic equivalent of the well-known *Dorfman-Steiner* relation follows, $0 \leqslant t \leqslant T$,

$$\frac{w^*(t)^{a(t)}}{p^*(t)\lambda^*(t)} = \frac{\Delta(t)}{\varepsilon(t)}. \tag{2.14}$$

Thus, at any time t, it is optimal to keep the ratio of the advertising expenditures and the revenue equal to the rate $\Delta(t)/\varepsilon(t)$; see Dorfman and Steiner (1954) for the static case and the original reference. If $\Delta(t) \equiv \Delta$ and $\varepsilon(t) \equiv \varepsilon$, it is optimal to keep this ratio constant all the time. Inventory and purchasing costs do not affect the *optimal ratio* of the expenditures on advertising to the revenue rate. Multiplying both sides of equation (2.14) with $e^{-R(t)}p^*(t)\lambda^*(t)$ and integrating from zero to T with respect to t, we obtain

$$\int_0^T e^{-R(t)} w^*(t)^{a(t)} dt = \int_0^T e^{-R(t)} \frac{\Delta(t)}{\varepsilon(t)} p^*(t)\lambda^*(t) dt.$$

The present value of the optimal *total* expenditures on advertising is a fraction of the present value of the optimally attainable revenue; recall, $\varepsilon(t) > 1 > \Delta(t)$. The verification of (2.14) is part of the proof of the following results which characterize the optimal control u^*.

Theorem 2.2.1 *Let $\varepsilon(t) \geqslant \underline{\varepsilon} > 1$, $0 \leqslant \delta(t) < a(t) \leqslant \overline{a}$, $\Delta(t) = \delta(t)/a(t)$, and $\gamma(t) = \frac{\varepsilon(t)-\Delta(t)}{1-\Delta(t)}$, $0 \leqslant t \leqslant T$. Let $c_w(t) := \left[\frac{\Delta(t)}{\varepsilon(t)-1}\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)}\right]^{\frac{1}{a(t)-\delta(t)}}$ and let $c_\lambda(t) := \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} c_w(t)^{\delta(t)}$. Let $\mu(t) > 0$ and let the demand follow $\lambda(t, p, w) = \mu(t)p^{-\varepsilon(t)}w^{\delta(t)}$. Let the cost rate $c(t)$ be given by (2.8). Then, the price function p^* and advertising rate*

w^* which satisfy (2.13) *for every t, $0 \leqslant t \leqslant T$, are given by*

$$p^*(t) = \frac{\varepsilon(t)}{\varepsilon(t)-1} c(t), \tag{2.15}$$

and

$$w^*(t) = c_w(t) \left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} \right)^{\frac{1}{a(t)-\delta(t)}}. \tag{2.16}$$

The sales rate $\lambda^*(t)$ and the profit margin $\nu^*(t)$ associated with the optimal price control (2.15) *and the optimal advertising control* (2.16) *are*[13]

$$\lambda^*(t) = c_\lambda(t) \left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-\Delta(t)}} \right)^{\frac{1}{1-\Delta(t)}} \overset{\delta(t)>0}{=} \frac{\varepsilon(t)-1}{\Delta(t)} \frac{w^*(t)^{a(t)}}{c(t)}, \tag{2.17}$$

and

$$\nu^*(t) = \frac{e^{-R(t)} c_\lambda(t)}{\gamma(t)-1} \left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} \right)^{\frac{1}{1-\Delta(t)}} = \frac{e^{-R(t)} c(t)}{\gamma(t)-1} \lambda^*(t) \overset{\delta(t)>0}{=} e^{-R(t)} \frac{1-\Delta(t)}{\Delta(t)} w^*(t)^{a(t)}. \tag{2.18}$$

Proof. For any t, the derivatives of the demand rate $\lambda(t,p,w)$ with respect to the control values are given by, $p, w > 0$,

$$\frac{\partial \lambda}{\partial p}(t,p,w) = -\frac{\varepsilon(t)}{p} \lambda(t,p,w), \qquad \text{and} \qquad \frac{\partial \lambda}{\partial w}(t,p,w) = \frac{\delta(t)}{w} \lambda(t,p,w);$$

the second derivatives are given by

$$\frac{\partial^2 \lambda}{\partial p^2}(t,p,w) = \frac{\varepsilon(t)(\varepsilon(t)+1)}{p^2} \lambda(t,p,w), \qquad \frac{\partial^2 \lambda}{\partial w^2}(t,p,w) = \frac{\delta(t)(\delta(t)-1)}{w^2} \lambda(t,p,w),$$

and

$$\frac{\partial^2 \lambda}{\partial p \partial w}(t,p,w) = -\frac{\delta(t)\varepsilon(t)}{pw} \lambda(t,p,w).$$

Since ν is differentiable in p and w, the (pointwise) first order conditions in the interior,

[13] By l'Hôpital's rule the formulas (2.17) and (2.18) are also justified for the case $\delta(t) = \Delta(t) = 0$, cf. Corollary 2.2.2.

i.e., $p > 0$ and $w > 0$, for p^* and w^* to be optimal imply

$$\begin{aligned}
\frac{\partial \nu}{\partial p}(t, p^*, w) &= e^{-R(t)} \left(\lambda(t, p^*, w) + (p^* - c(t)) \frac{\partial \lambda}{\partial p}(t, p^*, w) \right) \\
&= e^{-R(t)} \lambda(t, p^*, w) \left(1 - \varepsilon(t) \frac{p^* - c(t)}{p^*} \right) \stackrel{!}{=} 0, \qquad (2.19) \\
\frac{\partial \nu}{\partial w}(t, p, w^*) &= e^{-R(t)} \left((p - c(t)) \frac{\partial \lambda}{\partial w}(t, p, w^*) - a(t) w^{* a(t)-1} \right) \\
&= e^{-R(t)} \left(\frac{\delta(t)}{w^*} (p - c(t)) \lambda(t, p, w^*) - a(t) w^{* a(t)-1} \right) \stackrel{!}{=} 0. \qquad (2.20)
\end{aligned}$$

Solving (2.19) for $(p^* - c(t))$ and substituting this expression into the optimality condition (2.20) implies the *Dorfman-Steiner* relation (2.14). From (2.19) the expression of the optimal price policy (2.15) follows by elementary calculations. Inserting p^* into the *Dorfman-Steiner* relation and solving for w^* yields equation (2.16). The expressions for λ^* and ν^* follow by simply using the formulas of the optimal controls and by making again use of the *Dorfman-Steiner* relation and the definitions of the various parameters.

To see that p^* and w^* indeed satisfy (2.13), notice that p^* and w^* are unique solutions of the first order conditions. Moreover, since for the extreme cases $\lim_{p \to 0} \nu(t, p, w) < 0$ and $\lim_{p \to \infty} \nu(t, p, w) = -e^{-R(t)} w^{a(t)} \leqslant 0$, the value of p^* lies in the interior of the interval $(0, \infty)$. If $\delta(t) = 0$, it is optimal not to advertise at all; this observation goes in line with equation (2.16): if $\delta(t) = 0$, then $c_w(t) = 0$. Assuming a positive advertising elasticity $\delta(t)$, the demand at time t is zero if $w = 0$, i.e., $\nu(t, p, 0) = 0$. Since $a(t) > \delta(t)$ for all $t \in [0, T]$, it follows that $\lim_{w \to \infty} \nu(t, p, w) = -\infty$. Thus, the advertising expenditures exceed the revenue $p^*(t) \cdot \lambda^*(t)$, and an optimal advertising rate will be finite. Since $\mu(t) > 0$ and $c(t) > 0$ by assumption, the profit rate associated with the optimal price and advertising control is positive for any $t \geqslant 0$, i.e., $\nu^*(t) > 0$. Next, we check the sufficient conditions for optimality.

The second derivatives of the profit margin evaluated at the optimal policy values are:

$$\begin{aligned}
\frac{\partial^2 \nu}{\partial p^2}(t, p^*(t), w^*(t)) &= -(\varepsilon(t) - 1) e^{-R(t)} \frac{\lambda^*(t)}{p^*(t)} < 0, \\
\frac{\partial^2 \nu}{\partial w^2}(t, p^*(t), w^*(t)) &= -a(t)(a(t) - \delta(t)) e^{-R(t)} w^*(t)^{a(t)-2} < 0, \\
\frac{\partial^2 \nu}{\partial p \partial w}(t, p^*(t), w^*(t)) &= \delta(t) e^{-R(t)} \frac{\lambda^*(t)}{w^*(t)} \left(1 - \varepsilon(t) \frac{p^*(t) - c(t)}{p^*(t)} \right) = 0.
\end{aligned}$$

Obviously, the Hessian matrix of ν with respect to $p^*(t)$ and $w^*(t)$ is negative definite at any t. Thus, $p^*(t)$ and $w^*(t)$ are the (pointwise) global maximizers of $\nu(t)$ and are points in the interior of the set $\{(p,w)|p>0, w \geqslant 0\}$. ♦

Corollary 2.2.1 *The strategy u^* with $u^*(t) = (p^*(t), w^*(t))_{0 \leqslant t \leqslant T}$ defined in Theorem 2.2.1 is a feasible control which belongs to U_T. Thus,*

$$\pi_1^*(T) = \pi_1\left(T, u^*\right).$$

Proof. Since $\mu(t), c(t)$, and all other time-dependent functions are assumed to be piecewise continuous the controls p^* and w^* are also piecewise continuous, and ν^* is integrable on $[0,T]$. The restrictions on the parameters imply that $p^*(t) > c(t) > 0$ and $w^*(t) \geqslant 0$ throughout the interval $[0,T]$. The state equation (2.5) is satisfied, and the initial value x_0 is given by $x_0 = x^*(0) = \int_0^T e^{Q(s)}\lambda^*(s)ds$ according to (2.7). Since $u^*(t) = (p^*(t), w^*(t)) \in U_T$ and

$$\pi_1^*(T) = \sup_{u \in U_T} \pi_1(T,u) \leqslant \int_0^T \max_{p>0, w \geqslant 0} \{\nu(t,p,w)\}\, dt = \int_0^T \nu(t, p^*(t), w^*(t))dt = \pi_1\left(T, u^*\right)$$

the policies $p^*(t)$ and $w^*(t)$ maximize $\pi_1(T,p,w)$. ♦

The pointwise maximization of the integrand in (2.11) determines maximizers of the value of the integral. The optimal controls p^* and w^* only depend on the current point in time t and not on the time-to-go or the value of T: considering distinct time horizons $T_1 < T_2$, the optimal price and the optimal advertising rate will be the same for the interval $[0, T_1]$.[14] At any time t, $0 \leqslant t \leqslant T$, the optimal price depends only on the price elasticity $\varepsilon(t)$ and the cost function $c(t)$. In particular, it does neither depend on the arrival rate $\mu(t)$ nor on the advertising coefficients $a(t)$ and $\delta(t)$. Thus, the optimal price policy is the same whether the monopolist is allowed to advertise - or obliged to pay for promotion - or not! Equation (2.15) is the *common* static price formula of a monopolist which is also known as the Amoroso-Robinson relation: the marginal revenue $(\varepsilon(t)-1)/\varepsilon(t)p(t)$ must equal the (marginal) cost $c(t)$ at every point in time t.[15] The monopolist pursues a markup strategy where the size of the markup is determined by the term $\varepsilon(t)/(\varepsilon(t)-1)$. Therefore, the evolution of the price function depends on the price elasticity and the cost function. The optimal price trajectory starts at the value $\varepsilon(0)/(\varepsilon(0)-1)c_0$. Except for the special case $\ell(t) = q(t) = r(t) = 0$, $c(t)$

[14] Note, if the parameter functions depend on T, for example $\varepsilon(t) := \varepsilon(t,T) = \tilde{\varepsilon}e^{T-t}, \tilde{\varepsilon} > \underline{\varepsilon}, 0 \leqslant t \leqslant T$, i.e., the price sensitivity decreases towards T, the (optimal) policies will naturally depend on T.

[15] We multiply both sides of (2.15) by $(\varepsilon(t)-1)/\varepsilon(t)$.

is always increasing over time, cf. (2.9). Thus, the optimal price will increase whenever the elasticity is constant or is decreasing, i.e., it is optimal to (temporarily) run a market penetration strategy.[16] Whenever $\varepsilon(t)$ is increasing, there is no general statement on how the optimal price trajectory will evolve over time. To specify circumstances in which price skimming is optimal, i.e., prices decrease over time, we compute the derivative of the optimal price with respect to time[17]:

$$\dot{p}^*(t) = p^*(t)\left(\frac{\dot{c}(t)}{c(t)} - \frac{\dot{\varepsilon}(t)}{\varepsilon(t)(\varepsilon(t)-1)}\right) \overset{\varepsilon(t)\equiv\varepsilon}{=} \frac{\varepsilon}{\varepsilon-1}\dot{c}(t). \tag{2.21}$$

Hence, price skimming ($\dot{p}^* < 0$) will be optimal if and only if the relative increase in ε (divided by the term $\varepsilon - 1$) exceeds the relative increase of the cost function. In practice, both cases can be observed. Airlines typically raise their ticket prices towards the date of departure as business travelers are often last-minute bookers and less price-sensitive than tourists who buy their tickets weeks or months in advance. A price skimming strategy can often be observed when the freshness or up-to-dateness of the goods for sale is of concern: as the selling time progresses the customer's willingness to pay decreases for perishable products such as fruits or fashion, in other words: the customer's price sensitivity increases.

Another consequence of Theorem 2.2.1 is that if the price elasticity is nonincreasing, optimal prices might become very large and possibly tend to infinity if $T \to \infty$. This is the case because it is possible to set the price of the goods equal to any large value and still have positive demand (and thus a positive profit margin). In practical applications, one might often observe that $c_0 \gg \ell$, e.g., the value of a car in an exhibition room of a (luxury/classic) car retailer is relatively large compared to the costs of storing the car, and thus the relative price increase over time is very small. Although equation (2.15) gives the *exact* value of how to price the good(s), in such cases, a practitioner might be advised to keep the price constant over time (maybe at a value of $p^*(T/2)$ right from the beginning) and save the cost of relabeling prices. The cost of relabeling can be considered to be part of the carrying cost ℓ. In Section 2.3.2, we will examine the case of constant prices and dynamic advertising in more detail.

The optimal advertising strategy w^* depends on *all* model parameters. The ratio of the arrival intensity μ and the cost function c is of special importance: whenever the ratio $\mu(t)/c(t)^{\varepsilon(t)-1}$ is constant on some subinterval of $[0, T]$, it is optimal to keep the

[16] If the price elasticity is constant and $c(t)$ is constant, the special case $\ell(t) = q(t) = r(t) = 0$, then the optimal prices are also constant.

[17] All derivatives are assumed to exist. Equation (2.21) can be expressed in terms of elasticities with respect to time, cf. Section 2.4.1.

advertising level equal to $const^{\frac{1}{a(t)-\delta(t)}} \cdot c_w(t)$. Since the parameter function c_w only depends on the values of $a(t), \delta(t)$, and $\varepsilon(t)$, it is optimal to advertise at a constant level if the price elasticity and the adverting efficiency are also constant on the subinterval. Since $\varepsilon(t) > 1 > \Delta(t) \geqslant 0$, the terms $(\varepsilon(t) - 1)/\varepsilon(t))$ and $\Delta(t)/\varepsilon(t)$ are both less than one. Thus, it is obvious that the value of

$$c_w(t) = \left[\frac{\Delta(t)}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)} \right)^{\varepsilon(t)} \right]^{\frac{1}{a(t)-\delta(t)}} = \left[\frac{\Delta(t)}{\varepsilon(t)} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)} \right)^{\varepsilon(t)-1} \right]^{\frac{1}{a(t)-\delta(t)}}$$

is between 0 and 1. Moreover, in case $\mu(t) = c(t)^{\varepsilon(t)-1}$, it is optimal to advertise at a rate smaller than one, i.e., it is profitable for the monopolist to discourage potential buyers. In general, the optimal advertising rate, see (2.16), is larger than one if at time t the value of the arrival rate exceeds a term which depends on the cost rate, viz., $\Delta(t) > 0$,

$$\mu(t) > \frac{\varepsilon(t)-1}{\Delta(t)} \left(\frac{\varepsilon(t)}{\varepsilon(t)-1} \right)^{\varepsilon(t)} c(t)^{\varepsilon(t)-1} = \frac{\varepsilon(t)}{\Delta(t)} p^*(t)^{\varepsilon(t)-1}. \tag{2.22}$$

If inequality (2.22) is satisfied, the arrival intensity is at such a (high) level that it pays off to stimulate demand even more, i.e., to choose $w^*(t) > 1$ such that $w^*(t)^{\delta(t)} > 1$. In case the right-hand side of (2.22) is *too* large, it is optimal to actually curb the demand. This is the case if promotion is relatively expensive and has only a small effect, i.e., the advertising efficiency parameter $\Delta(t)$ is close to zero ($\delta(t) \ll a(t)$). From a different point of view the monopolist might argue that, if the costs are relatively high in relation to the arrival intensity μ, it does not pay off to attract more people. Hence, people should be discouraged of considering buying the retailer's goods. In practice, this might be achieved through a decreasing number of shop assistants or a reduction of their business hours as outlined above. In both cases the (labor) costs decrease; if these costs are (partly) assigned to promotions, see above, this decrease results in cutting back the promotion expenses.

We summarize the effect of advertising in the following two corollaries. The first one, Corollary 2.2.2, considers the case when advertising has no effect, i.e., $\delta(t) = 0$ for *every* point in time. The results are directly deduced from Theorem 2.2.1 and by allowing the value $\delta(t)$ for each time t to tend to zero. Naturally, it is optimal to set the advertising rate to zero, and the model becomes a pure pricing model. We state the optimal policies and associated values of this pure pricing model; we label these optimal values with the subscript "$_R$", a reference to the pure pricing model of Rajan et al. (1992). Corollary 2.2.3 compares the optimal policies and associated optimal values of the pure pricing model ($\delta(t) = 0$) with the optimal policies of the model where advertising has an effect ($\delta(t) > 0$). Although the optimal pricing strategy is not influenced by the value of the parameter δ, we are able to formulate conditions such that a retailer benefits from a market environment where promotion is effective.

Corollary 2.2.2 *Assume the conditions of Theorem 2.2.1 hold and let $\delta(t) \equiv 0$ for all $t \in [0, T]$. Then, the price p_R and advertising rate w_R which are determined by the right-hand side of (2.13) for every t, $0 \leqslant t \leqslant T$, are given by*

$$p_R(t) = \frac{\varepsilon(t)}{\varepsilon(t) - 1} c(t), \tag{2.23}$$

and

$$w_R(t) = 0. \tag{2.24}$$

The sales rate $\lambda_R(t)$ and the profit margin $\nu_R(t)$ associated with the price control (2.23) and the advertising control (2.24) are

$$\lambda_R(t) = \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)}}, \tag{2.25}$$

and

$$\nu_R(t) = \frac{e^{-R(t)}}{\varepsilon(t) - 1} \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}}. \tag{2.26}$$

Proof. To derive the formulas of Corollary 2.2.2 we use the formulas of Theorem 2.2.1 and let, for each t fixed, the value $\delta(t)$ tend to zero. Note, if δ tends to zero, then Δ also tends to zero.

To see (2.23), notice that the optimal price (2.15) does not depend on δ. Hence, $p_R(t) = p^*(t)$, $t \geqslant 0$. To see that $w_R(t) \equiv 0$ whenever $\delta(t)$ is zero, consider (2.16). For t fixed, the parameter c_w tends to zero if δ tends to zero. Hence, (2.24) follows.

To see that (2.25) holds, notice that

$$\begin{aligned}
c_\lambda(t) &= \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)} c_w(t)^{\delta(t)} \\
&= \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)} \left[\frac{\Delta(t)}{\varepsilon(t) - 1} \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)} \right]^{\frac{\delta(t)}{a(t) - \delta(t)}} \\
&= \left(\frac{\varepsilon(t) - 1}{\varepsilon(t)} \right)^{\varepsilon(t)\left(1 + \frac{\Delta(t)}{1 - \Delta(t)}\right)} \left(\frac{1}{\varepsilon(t) - 1} \right)^{\frac{\Delta(t)}{1 - \Delta(t)}} \Delta(t)^{\frac{\Delta(t)}{1 - \Delta(t)}}
\end{aligned}$$

converges to $\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)$ if $\Delta(t)$ goes to zero. Thus, $\lambda^*(t) = c_\lambda(t)\left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-\Delta(t)}}\right)^{\frac{1}{1-\Delta(t)}}$ converges to $\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)}} = \lambda_R(t)$. Formula (2.26) follows directly from equation (2.18), $\nu^*(t) = \frac{e^{-R(t)}c(t)}{\gamma(t)-1}\lambda^*(t)$, and the fact that $\gamma(t) = \frac{\varepsilon(t)-\Delta(t)}{1-\Delta(t)} \overset{\Delta(t)=0}{=} \varepsilon(t)$. ♦

Corollary 2.2.2 is a special case of Theorem 2.2.1 whenever δ equals zero. Moreover, Corollary 2.2.2 states the solution to the optimal pricing problem of Rajan et al. (1992) if the sales rate follows equation (2.4).

The scenario when δ equals zero is important. In some countries it is forbidden to promote pharmaceuticals, alcoholic drinks, or health care services.[18] The arrival rate μ in Corollary 2.2.2 can be interpreted to represent the arrival rate in a market where advertising is prohibited or where customers do not react to advertising. The discussion before Corollary 2.2.2 centered on the question whether sales are boosted by advertising at a given time point t, i.e., $\lambda^*(t) > \lambda_R(t)$, or not. Such a comparison assumes that the arrival rates and the price paths in both models are the same. By Corollary 2.2.2 we know that $p^* \equiv p_R$. Since $\lambda^*(t) = \mu(t)p^*(t)^{-\varepsilon(t)}w^*(t)^{\delta(t)}$ and $\lambda_R(t) = \mu(t)p_R(t)^{-\varepsilon(t)}$, the inequality $\lambda^*(t) > \lambda_R(t)$ is equivalent to $w^*(t) > 1$, see (2.22). Hence, if $w^*(t) > 1$, then the revenue rate $p^*(t)\lambda^*(t)$ at time t exceeds $p_R(t)\lambda_R(t)$. However, the model with active advertising incorporates the cost $w^*(t)^{a(t)}$. Therefore, it is not at all clear when advertising is beneficial at time t, i.e., whether $\nu^*(t) > \nu_R(t)$, or not. To compare both profit rates (and both sales rates), we assume that a positive arrival rate μ_R in the model without advertising is given, and the rate μ in the model with advertising is the product of μ_R and the reciprocal of a positive bounded function $\beta(t)$, i.e.,

$$\mu(t) = \frac{1}{\beta(t)}\mu_R(t). \tag{2.27}$$

Then, the demand rates of both models do not only differ by the factor $w^*(t)^{\delta(t)}$ but also by the factor $\beta(t)$. Since $p^* \equiv p_R$ the optimal sales rate in the model without advertising equals $\lambda_R(t) = \mu_R(t)p_R(t)^{-\varepsilon(t)} = \mu(t)p^*(t)^{-\varepsilon(t)}\beta(t)$. Thus, the factor $\beta(t)$ *replaces* the factor $w^*(t)^{\delta(t)}$ in the sales rate λ^* associated with the model with advertising. In contrast to the advertising cost rate $w^*(t)^{a(t)}$ the factor $\beta(t)$ incurs no cost; β acts like a *free* promotion rate.

At the beginning of this section we argued that the function μ may have a product form where each factor has its own economic interpretation. For the product (2.27) we

[18]For instance, Example 11.1 in Mansfield (1994), p. 344, considers the effect of a ban on advertising of eyeglasses in some states of the US.

offered a possible interpretation of the factor μ_R, viz. μ_R is the arrival intensity of a market where advertising has no effect. The factor $\frac{1}{\beta(t)}$ can then be interpreted to be a (modified) response constant, cf. Appendix 2.

Corollary 2.2.3 *Let $\mu(t) > 0$ and let p^*, w^*, λ^*, and ν^* be given by Theorem 2.2.1. Let $\beta(t) > 0$ and let p_R, λ_R, and ν_R be given by Corollary 2.2.3 when the arrival intensity is $\mu_R(t) := \beta(t)\mu(t), t \geqslant 0$, ceteris paribus.*

(i) Then, $t \geqslant 0$,

$$p^*(t) = p_R(t), \tag{2.28}$$

$$\lambda^*(t) = \frac{w^*(t)^{\delta(t)}}{\beta(t)}\lambda_R(t), \tag{2.29}$$

and

$$\nu^*(t) = \frac{1-\Delta(t)}{\beta(t)}w^*(t)^{\delta(t)}\nu_R(t). \tag{2.30}$$

(ii) For any $t \geqslant 0$, if $w^*(t)^{\delta(t)} > \beta(t)$, then

$$\lambda^*(t) > \lambda_R(t). \tag{2.31}$$

If $w^*(t)^{\delta(t)} < \beta(t)$, the inequality (2.31) *is reversed; if $w^*(t)^{\delta(t)} = \beta(t)$, then $\lambda^*(t) = \lambda_R(t)$*

(iii) For any $t \geqslant 0$, if $w^*(t)^{\delta(t)} > \frac{\beta(t)}{1-\Delta(t)}$, then

$$\nu^*(t) > \nu_R(t). \tag{2.32}$$

If $w^*(t)^{\delta(t)} < \frac{\beta(t)}{1-\Delta(t)}$, the inequality (2.32) *is reversed; if $w^*(t)^{\delta(t)} = \frac{\beta(t)}{1-\Delta(t)}$, then $\nu^*(t) = \nu_R(t)$.*

Proof. (*i*) Equation (2.28) follows from equations (2.15) and (2.23). To see (2.29) and (2.30) we rely on Theorem 2.2.1 and Corollary 2.2.2 and the following calculations.

Step 1: If $\delta(t) > 0$, the optimal profit rate and the optimal advertising rate are given by, see Theorem 2.2.1, $t \geqslant 0$,

$$\lambda^*(t) = \frac{\varepsilon(t)-1}{\Delta(t)}\frac{w^*(t)^{a(t)}}{c(t)},$$

and

$$\nu^*(t) = \frac{e^{-R(t)}c_\lambda(t)}{\gamma(t)-1}\left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}}\right)^{\frac{1}{1-\Delta(t)}} = \frac{e^{-R(t)}c(t)}{\gamma(t)-1}\lambda^*(t) = e^{-R(t)}\frac{1-\Delta(t)}{\Delta(t)}w^*(t)^{a(t)}.$$

Moreover, the optimal advertising rate is given by (2.16),

$$w^*(t) = c_w(t)\left(\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}}\right)^{\frac{1}{a(t)-\delta(t)}} = \left[\frac{\Delta(t)}{\varepsilon(t)-1}\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)}\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}}\right]^{\frac{1}{a(t)-\delta(t)}}.$$

Step 2: If $\delta(t) \equiv 0$ and when the arrival rate is μ_R, the optimal sales rate and the optimal profit rate are given by, see Corollary 2.2.2,

$$\begin{aligned}\lambda_R(t) &= \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)}\frac{\mu_R(t)}{c(t)^{\varepsilon(t)}}, \quad \text{and}\\ \nu_R(t) &= \frac{e^{-R(t)}}{\varepsilon(t)-1}\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)}\frac{\mu_R(t)}{c(t)^{\varepsilon(t)-1}}.\end{aligned}$$

Step 3: Since $\mu(t) = \frac{\mu_R(t)}{\beta(t)}$, elementary calculations show that

$$\begin{aligned}w^*(t)^{a(t)-\delta(t)} &= \frac{\Delta(t)}{\varepsilon(t)-1}\left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)}\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} = \frac{\Delta(t)}{\varepsilon(t)-1}\frac{c(t)}{\beta(t)}\lambda_R(t)\\ &= \frac{\Delta(t)}{\beta(t)}e^{R(t)}\nu_R(t).\end{aligned} \tag{2.33}$$

Hence,

$$\begin{aligned}\lambda^*(t) &= \frac{\varepsilon(t)-1}{\Delta(t)}\frac{w^*(t)^{a(t)}}{c(t)} = \frac{\varepsilon(t)-1}{\Delta(t)}\frac{w^*(t)^{\delta(t)}}{c(t)}w^*(t)^{a(t)-\delta(t)}\\ &= \frac{\varepsilon(t)-1}{\Delta(t)}\frac{w^*(t)^{\delta(t)}}{c(t)}\frac{\Delta(t)}{\varepsilon(t)-1}\frac{c(t)}{\beta(t)}\lambda_R(t) = \frac{w^*(t)^{\delta(t)}}{\beta(t)}\lambda_R(t)\end{aligned}$$

yields (2.29). Similarly, we show equation (2.30) holds true:

$$\begin{aligned}\nu^*(t) &= e^{-R(t)}\frac{1-\Delta(t)}{\Delta(t)}w^*(t)^{a(t)} = e^{-R(t)}\frac{1-\Delta(t)}{\Delta(t)}w^*(t)^{\delta(t)}w^*(t)^{a(t)-\delta(t)}\\ &= e^{-R(t)}\frac{1-\Delta(t)}{\Delta(t)}w^*(t)^{\delta(t)}\frac{\Delta(t)}{\beta(t)}e^{R(t)}\nu_R(t) = \frac{1-\Delta(t)}{\beta(t)}w^*(t)^{\delta(t)}\nu_R(t).\end{aligned}$$

(ii) If, for any $t \geqslant 0$, $w^*(t)^{\delta(t)} > \beta(t)$, then it follows immediately from (2.29) that

$$\lambda^*(t) = \frac{w^*(t)^{\delta(t)}}{\beta(t)}\lambda_R(t) > \lambda_R(t);$$

if $w^*(t)^{\delta(t)} < \beta(t)$, then the inequality $\lambda^*(t) < \lambda_R(t)$ follows. If $w^*(t)^{\delta(t)} = \beta(t)$, then the equality $\lambda^*(t) = \lambda_R(t)$ follows.

(iii) If, for any $t \geqslant 0$, $w^*(t)^{\delta(t)} > \frac{\beta(t)}{1-\Delta(t)}$, then the inequality $\nu^*(t) > \nu_R(t)$ follows from (2.30). If we assume $w^*(t)^{\delta(t)} < \frac{\beta(t)}{1-\Delta(t)}$, then (2.30) implies $\nu^*(t) < \nu_R(t)$. If $w^*(t)^{\delta(t)} = \frac{\beta(t)}{1-\Delta(t)}$, then $\nu^*(t) = \nu_R(t)$ ♦

In the following, we apply Corollary 2.2.3 on the whole interval $[0, T]$. Corollary 2.2.3 enables us to compare two markets in terms of the optimal price, the sales rate, and the profit margin: one market, where advertising has an effect, i.e., δ is positive on the whole interval $[0, T]$, and another market, where advertising has no effect, i.e., $\delta = 0$ on $[0, T]$. From the consumer's perspective it is important to note that the (optimal) price is the same in both markets. Hence, if the optimal advertising rate is high enough ($w^*(t)^{\delta(t)} > \beta(t)$) in the market where advertising has an effect, more customers will buy the product: consumers are not disadvantaged by advertising. The monopolist attracts more (potential) buyers and hence increases her revenue. However, since advertising also incurs costs, the optimal advertising rate must be larger than $(\beta(t)/(1-\Delta(t)))^{1/\delta(t)}$ to guarantee that the monopolist benefits from advertising at time t, i.e., the net profit rate is larger in the market with advertising than in the market without. In contrast to that, it is hard to imagine practical situations where the monopolist is doing better in a market where customers are not receptive to advertising compared to a market where advertising is effective. However, statutory regulations may prohibit advertising and put the company into a position where the additional benefits from advertising are unmarketable. Note, the profit rate of the monopolist is always strictly greater than zero - independent of the influence of advertising, i.e., $\nu^*(t) > 0$ and $\nu_R(t) > 0$ for all $t \geqslant 0$.

Right now, we postpone a more detailed analysis of the dynamic price-advertising model to Section 2.4. There, we provide illustrations and numerical examples to examine the dynamic price-advertising model at length. Comparisons between the dynamic price-advertising model and the pure pricing model will also be part of the analysis in Chapter 4, where we compare the pure pricing model and the model that allows advertising with respect to the (average) optimal cycle length, cf Theorem 4.2.4. In the following section, we consider the (partially) static models where only one of the controls, the price or the advertising rate, are supposed to be dynamic and the remaining control has to be set to a fixed value throughout the whole sales period.

2.3 Combinations of Dynamic and Static Controls

2.3.1 Dynamic Pricing but Advertising Rates are Constant

Corollary 2.2.3 quantifies the differences of the following two markets: one market where advertising has no effect and another one where advertising is effective. But the two distinct market settings allow for a second interpretation. When no advertising term is explicitly included in the demand function, cf. the pure pricing model, the promotional influence is captured by the term μ_R. Although we assume that this promotion comes for free, one can still evaluate the cost of the free (constant) advertising rate.

In such cases the associated total advertising costs are fixed for a cycle of length T and can be neglected when solving the optimization problem (the fixed cost actually provides a lower bound for the value of the monopolist's profit as the business operations must at least compensate for this expense). It seems reasonable to include these costs but give the retailer the opportunity to choose the level on her own. Alternatively, a company might be committed to the advertising level it chooses and is not allowed to change this total effort afterwards. Naturally, the profit will be smaller than the profit in the dynamic setting, except for the case if $\mu(t) \equiv c(t)$ and price and advertising elasticities are constant, see condition (2.22); in this special case profit rates are equal. But what is the benefit from being able to choose the advertising rate dynamically? One can also ask: if the monopolist gets free promotion in the pure pricing model, which amount should be charged to compensate for the external promotion effect captured by μ?

Since we are interested in a constant advertising rate, it makes sense to keep the corresponding parameters also constant over time. In the following, we will consider the advertising elasticity δ and the cost coefficient a to be constant, i.e., $\delta(t) \equiv \delta$, and $a(t) \equiv a < \bar{a}$. Thus, the parameter of advertising efficiency is also constant, i.e., $\Delta(t) \equiv \Delta$. Notice that we still allow for a time-dependent price elasticity $\varepsilon(t) > \underline{\varepsilon}$. The analysis in the previous section, where we assume dynamic price and dynamic advertising, shows that the optimal price policy p^* does not depend on the advertising rate.[19] Hence, it is feasible to apply a two-stage approach. First, we solve for the optimal (dynamic) pricing policy which - as expected - is identical to p^*. Second, we identify the

[19] Advertising at rate zero implies zero profit as long as $\delta(t) > 0$, i.e., the price is irrelevant since no customers will be attracted and nothing will be sold.

optimal (constant) advertising rate which will now depend on the cycle length T. The state equation (2.5) remains valid, but the control u now consists of a (dynamic) pricing scheme and a constant advertising level throughout $[0,T]$, i.e., $u(t) = (p(t), w) \in \bar{U}_T$, where

$$\bar{U}_T = \left\{ u \;\middle|\; \begin{array}{l} u(t) = (p(t), w)\,, \text{ and } u(t) \text{ is a vector-valued piecewise continuous} \\ \text{function on } [0,T], 0 \leqslant t \leqslant T, \text{ such that the solution of (2.5) with} \\ \text{respect to } x(t) \text{ is uniquely determined and all integrals to be} \\ \text{encountered in the sequel exist; moreover, } p(t) > 0, w \geqslant 0 \end{array} \right\}.$$

Note, $\bar{U}_T \subset U_T$. Let $M(T, p(\cdot))$ denote the net present value of the *price contribution* to the total profit of a given price strategy in the time interval $[0,T]$,

$$M(T, p(\cdot)) := \int_0^T e^{-R(t)} \left[(p(t) - c(t))\, \mu(t) p(t)^{-\varepsilon(t)} \right] dt. \tag{2.34}$$

Whenever a pricing policy has the property $p(t) > c(t)$, $0 \leqslant t \leqslant T$, then $M(T, p(\cdot)) > 0$; in particular, this is true, for the optimal pricing policy p^*, cf. (2.15). We call $M(T, p(\cdot))$ the price contribution as it equals the present value of the net revenue (without advertising cost) if the advertising rate equals one. Let $D(T)$ denote the present value of a "payment-flow-of-rate-one" received from presence until time T, i.e.,

$$D(T) := \int_0^T \xi(s) ds = \int_0^T e^{-R(s)} ds, \tag{2.35}$$

where $\xi(s)$ is the discount factor, cf. the remarks on (hyperbolic) discounting in Section 2.1. Note, in case of a constant discount rate $r(t) \equiv r$, we have $D(T) = \frac{1-e^{-rT}}{r} \underset{r \to 0}{\longrightarrow} T$.

For any feasible policy $u \in \bar{U}_T$ we can rewrite the objective function as follows:

$$\begin{aligned} \pi_1(T, u) &= \int_0^T \nu(t, u) dt = \int_0^T e^{-R(t)} \left[(p(t) - c(t))\, \mu(t) p(t)^{-\varepsilon(t)} w^{\delta} - w^{a} \right] dt \\ &= w^{\delta} \int_0^T e^{-R(t)} (p(t) - c(t))\, \mu(t) p(t)^{-\varepsilon(t)} - w^{a} \int_0^T e^{-R(t)} dt \\ &= w^{\delta} M(T, p(\cdot)) - w^{a} D(T), \end{aligned}$$

where the last line states the objective function in terms of the total net present revenue minus the net present cost (of advertising). The *leverage* effect of advertising becomes evident once more: $M(T, p(\cdot))$, the net present value of the pricing policy, is enhanced if $w > 1$ and reduced if $w < 1$. If $w = 1$ (or $\delta = 0$), the total net revenue is equivalent to the one of the pure pricing model. Another implication of this last line is that if $M(T, p(\cdot)) > 0$, there exists a positive advertising rate such that the profit will be positive. If $M(T, p(\cdot)) \leqslant 0$, it is optimal to set $w \equiv 0$ and make zero profit. However, we will show that since it is always possible to choose a price policy such that $p(t) > c(t)$ for all $t \in [0, T]$, the function $M(T, p(\cdot))$ and thus the associated profit function $\pi_1(T, u)$ will always take positive values with an appropriate choice of w.

In the following, we will label results and expression related to the optimal solution in case of a constant advertising rate by a " $^-$ " superscript. Let $\bar{u}$, where $\bar{u}(t) = (\bar{p}(t), \bar{w}_T)$, denote the strategy that maximizes $\pi_1(T, u)$, and let $\bar{\pi}_1(T) := \pi_1(T, \bar{u})$.[20] Since the value of the integral expression $M(T, p(\cdot))$ does not depend on the choice of w, the aforementioned two-stage problem to be solved is

$$\bar{\pi}_1(T) = \sup_{u \in \bar{U}_T} \pi_1(T, u) = \sup_{w \geqslant 0} \left\{ w^{\delta} \sup_{p(\cdot) > 0} \{M(T, p(\cdot))\} - w^a D(T) \right\}. \tag{2.36}$$

The inner maximization problem in (2.36) provides the optimal pricing strategy by maximizing the integrand of $M(T, p(\cdot))$ with respect to the price for every t. We will show that, if the optimal price policy is used, $M(T, \bar{p}(\cdot))$ is indeed positive. Similar to the dynamic problem in Section 2.2, the Amoroso-Robinson relation holds pointwise for the optimal price at time t. Given the optimal price, the outer maximization problem can be solved and determines the best constant advertising rate. We will show that this approach is feasible in the proof of the following result.

Theorem 2.3.1 *Assume* $0 \leqslant \delta < a < \overline{a}$, $\Delta = a/\delta$, *and* $\varepsilon(t) > \underline{\varepsilon} > 1$ *for all* $t \in [0, T]$. *Let* $D(T) = \int_0^T e^{-R(t)} dt$, *and* $M(T, p(\cdot)) = \int_0^T e^{-R(t)} \left[(p(t) - c(t)) \mu(t) p(t)^{-\varepsilon(t)} \right] dt$. *Assume* $\mu(t) > 0$*; let the demand follow* $\lambda(t, p, w) = \mu(t) p^{-\varepsilon(t)} w^{\delta}$, *and let the cost rate* $c(t)$ *be given by* (2.8). *Then, the dynamic price* $\bar{p}(t), t \geqslant 0$, *and the constant advertising rate* $\bar{w}_T$ *which solve* (2.36) *are given by*

$$\bar{p}(t) = p^*(t) = \frac{\varepsilon(t)}{\varepsilon(t) - 1} c(t), \tag{2.37}$$

[20] The existence and finiteness of the optimal value and the existence and uniqueness of the optimal control are proved below.

and

$$\bar{w}_T = \left(\Delta \frac{\bar{M}(T)}{D(T)}\right)^{\frac{1}{a-\delta}}, \tag{2.38}$$

where $\bar{M}(T) := M(T, \bar{p}(\cdot)) = \int\limits_0^T e^{-R(t)} \frac{1}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} dt.$

Proof: To solve the inner maximization problem in (2.36) we take the derivative of the integrand of $M(T, p(\cdot))$ with respect to p for every t. Then, the first order condition for the optimal price $\bar{p}$ at time t, $0 \leqslant t \leqslant T$, implies that

$$e^{-R(t)}\mu(t)\left(\bar{p}^{-\varepsilon(t)} - \varepsilon(t)\left(\bar{p} - c(t)\right)\bar{p}^{-\varepsilon(t)-1}\right) = 0,$$

and equation (2.37) follows directly by elementary calculations. One can easily show that the integrand of $M(T, p(\cdot))$ is strictly concave in p at every point in time t as long as $p < (\varepsilon(t)+1)/(\varepsilon(t)-1)c(t)$, and thus for $\bar{p}$. Note, $\bar{p}(t) > c(t)$ for every point in time t. Hence, $\bar{M}(T) = M(T, \bar{p}(\cdot))$ is positive, and the optimal advertising constant will be positive too. Thus, the outer maximization problem in (2.36) can be solved by applying the first order condition, i.e., $\bar{w}_T$ must satisfy

$$\delta \bar{w}_T^{\delta-1}\bar{M}(T) - a\bar{w}_T^{a-1}D(T) = 0.$$

Formula (2.38) follows by solving for $\bar{w}_T$. The second order condition shows that $\bar{w}_T$ is indeed the profit maximizing advertising rate when the optimal pricing scheme satisfies $M(T, \bar{p}(\cdot)) > 0$. To this end, notice that

$$\begin{aligned}
\frac{\partial^2 \pi_1}{\partial w^2}(T, \bar{p}(\cdot), \bar{w}_T) &= \delta(\delta-1)\bar{w}_T^{\delta-2}M(T, \bar{p}(\cdot)) - a(a-1)\bar{w}_T^{a-2}D(T) \\
&\overset{a \neq 1}{=} a(a-1)\bar{w}_T^{a-2}D(T)\left(\frac{\delta-1}{a-1}\Delta\bar{w}_T^{\delta-a}\frac{\bar{M}(T)}{D(T)} - 1\right) \\
&= a(a-1)\bar{w}_T^{a-2}D(T)\left(\frac{\delta-1}{a-1} - 1\right) \\
&= -a(a-\delta)\bar{w}_T^{a-2}D(T) \\
&< 0,
\end{aligned}$$

since $\delta < a$ and $D(T) > 0$. To see that the second derivative is negative in the case $a \equiv 1$, notice that the first line immediately implies this fact. Thus, $\bar{w}_T$ satisfies the first and the second order conditions.

The price policy $\bar{p}(t)$ is piecewise continuous since it is a composition of two piecewise continuous functions; moreover, $\bar{w}_T$ is a positive constant. Thus, the integral in (2.3.1) is well defined for $\bar{u}$, where $\bar{u}(t) = (\bar{p}(t), \bar{w}_T)$, and the state equation (2.5) is satisfied with $\bar{x}_0 = \bar{x}(0) = \int_0^T e^{Q(s)} \lambda(s, \bar{u})\, ds$. Since $\bar{u}(t) \in \bar{U}_T$, it follows that

$$\bar{\pi}_1(T) = \sup_{u \in \bar{U}_T} \pi_1(T, u) \leqslant \max_{w>0} \max_{p(\cdot)>0} \left\{ \pi_1(T, p(\cdot), w) \right\} = \pi_1(T, \bar{u}). \tag{2.39}$$

Hence, $\bar{u}$ maximizes $\pi_1(T, u)$ in all these cases where only static advertising policies are allowed. ♦

Whether we consider a dynamic advertising rate or a static one, the optimal pricing schemes are the same. The Amoroso-Robinson relation also holds if advertising has no effect ($\delta \equiv 0$). The optimal rate $\bar{w}_T$ depends on all parameters. Similar to the dynamic case the ratio of the arrival intensity μ and the cost functional $c(t)$ (to the power of $\varepsilon(t) - 1$) is most important since it is part of the integrand of $M(T)$. The denominator $D(T)$ can be thought of as a *time averaging* component taking discounting into account. Thus, the optimal constant advertising rate $\bar{w}_T$ can be interpreted as the (time adjusted) average of the dynamic rate. To be precise, it can easily be shown that

$$\int_0^T e^{-R(s)} \bar{w}_T^{a-\delta} ds \;=\; D(T)\, \bar{w}_T^{a-\delta} \;=\; \int_0^T e^{-R(s)} w^*(s)^{a-\delta} ds. \tag{2.40}$$

The present value of the constant rate $\bar{w}_T^{a-\delta}$ and the present value of the dynamic rate $w^*(t)^{a-\delta}$, the right-hand side of (2.40), are equal. Without discounting the expression $D(T)$ equals T and the constant rate can be calculated by averaging the corresponding accumulated dynamic rate; rewrite the relation (2.40) as $\bar{w}(T) = (\int_0^T w^*(t)^{a-\delta} dt/T)^{1/(a-\delta)}$. Comparing the dynamic rate with the constant one at a specific point t, one easily finds that $w^*(t) > \bar{w}_T$ if and only if

$$\frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} > \frac{\int_0^T e^{-R(s)} \frac{\mu(s)}{c(s)^{\varepsilon(s)-1}} ds}{D(T)}. \tag{2.41}$$

Thus, if the ratio of the arrival rate and the cost rate (to the power of $\varepsilon(t) - 1$) at some point t is larger (smaller) than the average value of this ratio at present cost, then the optimal dynamic advertising rate lies above (below) the optimal constant rate. Since the cost rate increases over time, the dynamic rate will typically be larger at the beginning of the interval $[0, T]$ and will fall below the constant rate at the end of the cycle. A property that follows directly is that the optimal sales rates in both models -

the dynamic advertising model and the static advertising model - behave according to the advertising rates since prices are identical. The sales rates of both models only differ in the promotion component. Before we compare the sales rates and the profit rates of the dynamic model with the rates of the static one we first give explicit formulas.

Corollary 2.3.1 *Let the assumptions of Theorem* (2.3.1) *be satisfied. Then, the sales rate* $\bar{\lambda}$ *and the profit margin* $\bar{\nu}$ *associated with the (dynamic) price control* (2.37) *and the (constant) advertising control* (2.38) *are given by*

$$\bar{\lambda}(t) := \lambda(t, \bar{u}(t)) = \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)}} \left(\Delta \frac{\bar{M}(T)}{D(T)}\right)^{\frac{\Delta}{1-\Delta}}, \tag{2.42}$$

and

$$\bar{\nu}(t) := \nu(t, \bar{u}(t)) = e^{-R(t)} \bar{w}_T^a \left(\frac{1}{\Delta} \frac{1}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{D(T)}{\bar{M}(T)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} - 1\right). \tag{2.43}$$

The profit of one cycle of length T *associated with the optimal control* $\bar{u} \in \bar{U}_T$ *equals*

$$\bar{\pi}_1(T) = \int_0^T \bar{\nu}(t) dt = \frac{1-\Delta}{\Delta} D(T) \bar{w}_T^a. \tag{2.44}$$

Proof: The formulas of Corollary 2.3.1 follow by evaluating the sales rate and the profit margin for the optimal control $\bar{u}$ described in Theorem 2.3.1 and by simple algebra. To derive formula (2.44), we make use of $D(T) = \int\limits_0^T e^{-R(t)} dt$ and the optimal price contribution $\bar{M}(T) = M(T, \bar{p}(\cdot)) = \int\limits_0^T e^{-R(t)} \frac{1}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} dt$:

$$\begin{aligned}
\int_0^T \bar{\nu}(t) dt &= \int_0^T e^{-R(t)} \bar{w}_T^a \left(\frac{1}{\Delta} \frac{1}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{D(T)}{\bar{M}(T)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} - 1\right) dt \\
&= \frac{\bar{w}_T^a}{\Delta} \frac{D(T)}{\bar{M}(T)} \int_0^T e^{-R(t)} \frac{1}{\varepsilon(t)-1} \left(\frac{\varepsilon(t)-1}{\varepsilon(t)}\right)^{\varepsilon(t)} \frac{\mu(t)}{c(t)^{\varepsilon(t)-1}} dt - \bar{w}_T^a \int_0^T e^{-R(t)} dt \\
&= \frac{\bar{w}_T^a}{\Delta} \frac{D(T)}{\bar{M}(T)} \bar{M}(T) - \bar{w}_T^a D(T) \\
&= \frac{1-\Delta}{\Delta} D(T) \bar{w}_T^a.
\end{aligned}$$

♦

These results go in line with the observations made in the dynamic advertising case. The optimally controlled sales rate co-moves with the value of $\mu(t)/c(t)^{\varepsilon(t)}$. For instance, in the case of a constant arrival intensity ($\mu(t) \equiv \mu$) the associated sales rate decreases over time. The maximized profit is a fraction of the present value of the (constant) advertising cost flow over the cycle. The value of the advertising efficiency parameter Δ determines whether the factor $(1-\Delta)/\Delta$ acts as a markup or as a markdown on the (now total) advertising costs. Note, due to the monopolist's constraint to fix the advertising rate, the *Dorfman-Steiner* relation, see equation (2.14), only holds on *average*. Thus, the *advertising-spending-to-revenue-ratio* is not fixed and can even become larger than one at some time $t \in [0,T]$, i.e., the monopoly is temporarily running deficits. The parameter restrictions on $\Delta(t)$ and $\varepsilon(t)$ ensure that this will never be the case in the dynamic setup, i.e., $\nu^*(t) > 0$ for all $t \in [0,T]$, see Theorem 2.2.1. As shown above we can guarantee that the retailer will make positive total profits over the cycle. It is however possible that the profit rate is negative for some interval(s) in $[0,T]$. Formula (2.43) reveals *when* this will be the case. Rewriting the difference in equation (2.43) and making use of relation (2.40) one obtains that this difference - and thus $\bar{\nu}(t)$ - will be negative if $w^*(t)/(T\bar{w}_T) < \Delta^{1/(a-\delta)}$. The profit rate is maximized pointwise in the dynamic case and the optimal price strategies are identical in both settings. The maximized profit rate associated with dynamic advertising will never be smaller than the one associated with an optimal constant advertising rate. A comparison of the total profit per cycle in both scenarios shows that the gain due to the opportunity to dynamically control the advertising level can be written as

$$\pi_1^*(T) - \bar{\pi}_1(T) = \frac{1-\Delta}{\Delta} \int_0^T e^{-R(t)} \left(w^*(t)^a - \bar{w}(T)^a\right) dt.$$

Thus, the benefit of dynamic advertising equals the factor $\frac{1-\Delta}{\Delta}$ times the present value of the difference in the total advertising spending.

Since $c(t)$ is a nondecreasing function we are able to derive bounds for the optimal one-cycle profit in the time-homogeneous case.

Proposition 2.3.1 *In the time-homogeneous setting, i.e., $a(t) \equiv a, \delta(t) \equiv \delta, \varepsilon(t) \equiv \varepsilon$, $\mu(t) \equiv \mu$, and if $r(t) \equiv 0$, $0 \leqslant t \leqslant T$, the optimal profit $\pi_1^*(T)$ associated with dynamic advertising (and dynamic pricing) and the optimal profit $\bar{\pi}_1(T)$ associated with static advertising (and dynamic pricing) satisfy the following inequalities:*

$$\frac{1-\Delta}{\Delta} {c_w}^a \left(\frac{\mu}{c(T)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} T \leqslant \bar{\pi}_1(T) \leqslant \pi_1^*(T) \leqslant \frac{1-\Delta}{\Delta} {c_w}^a \left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} T. \tag{2.45}$$

If, in addition, the inventory cost and the decay rate are both zero, i.e., $\ell(t) = q(t) = 0$*, then all inequality signs of* (2.45) *will become equal signs.*

Proof: Choosing a constant advertising rate is a feasible policy in the problem where dynamic advertising is allowed. Hence, the optimal profit associated with dynamic advertising can not be smaller than the optimal profit associated with the constant advertising rate, i.e., $\bar{\pi}_1(T) \leqslant \pi_1^*(T)$. The properties $r(t) \equiv 0$ and $\mu(t) \equiv \mu$, $0 \leqslant t \leqslant T$, imply $D(T) = T$ and

$$\bar{\pi}_1(T) = \frac{1-\Delta}{\Delta} {c_w}^a \left(\frac{\mu}{T}\right)^{\frac{1}{1-\Delta}} T \left(\int_0^T \frac{1}{c(t)^{\varepsilon-1}} dt\right)^{\frac{1}{1-\Delta}}. \tag{2.46}$$

Since $c(t)$ is a nondecreasing function, the integral on the right-hand side of (2.46) is bounded from below by $T/(c(T)^{\varepsilon-1}))$, and it is bounded from above by $T/(c_0^{\varepsilon-1}))$. Hence, the first inequality in (2.45) follows for $\bar{\pi}_1(T)$. The corresponding analysis of $\pi_1^*(T)$ provides identical bounds.

In case $\ell(t) = q(t) = 0$ we have $c(t) = c_0$, $0 \leqslant t \leqslant T$, and the upper bound and lower bound are the same. ♦

The results of Proposition 2.3.1 are of limited value if the bounds are far apart. However, the proposition offers valuable insights in the following case: if the running costs are *small* relative to the unit cost, i.e., $\ell \ll c_0$, the lower and the upper bounds on the profits are close. This effect is amplified by a *small* ε value (close to one) and a small Δ value (close to zero). For instance, if $\varepsilon = 2, \Delta = 0.3, T = 10$, and the inventory cost per time unit amounts to five percent of the unit cost, i.e., $l = 0.05c0$, the lower bound accounts for approximately 56 percent of the upper bound. If the inventory cost per unit and period is only one percent of the unit cost, the lower bound accounts for approximately 87 percent of the upper bound; if $\ell = 0.001c_0$, this value increases to 99 percent. Large differences between inventory cost and unit cost are not unusual in *high-priced* retailing, for instance, in car retailing, in the jewelry business or in the art business. We will give additional illustrations and examples in Section 2.4. There, we will also analyze the results of this section in greater detail.

In the following section we will examine the case that is still missing in our analysis: constant pricing but dynamic advertising.

2.3.2 Dynamic Advertising but Prices are Constant

In the previous sections we considered dynamic pricing. The results suggest synchronizing prices with cost rates. Since the cost rate monotonically increases over time, except for some very special cases, optimal dynamic prices should also increase over time. In E-commerce, for example, when selling products on the Internet or selling Internet based applications, the retailer is able to change prices continuously. Although technological progress makes it possible to implement dynamic pricing in classic retailing, it is far from being standard. If no electronic system is put in place that allows to adjust prices, every price change induces price setting costs. Especially, when the price tags of many different items have to be changed, as in grocery stores or supermarkets, these costs can become large. Although the demand function that we consider does not account for such factors as price changes or price persistence, customers become usually suspicious when they observe regularly changing prices. In the sequel, we will consider the extreme case of no price changes, i.e., a constant price is chosen throughout the cycle of length T.[21] However, the advertising rate is supposed to be dynamically controlled.

A prominent field of application of the fixed-price-dynamic-advertising setting is the so-called *variety shops* or *dollar stores*. The business concept dictates prices to be fixed, i.e., the items are sold at one common price or a few distinct prices only. Except for choosing the price category (one dollar, two dollars,...) the option to vary prices does not exist. However, advertising activities, either by a wholesaler who promotes many stores of the same brand or by an individual shop owner, that might vary over time play a prominent role for the business concept. In general, customers will encounter a mixture of both marketing tools. Dollar stores heavily rely on advertising as they usually sell items that are lacking natural demand and bear no particular characteristics except the common price.

Like in the previous section, it makes sense to assume a time-independent price elasticity $\varepsilon(t) = \varepsilon > \underline{\varepsilon} > 1$, but to allow for time-dependent parameters $a(t)$ and $\delta(t)$. We will label expressions related to this setting - especially the optimal control and values associated with the optimal control - by a "~" superscript.

We denote by $\tilde{U}_T$ the set of feasible controls, i.e.,

$$\tilde{U}_T = \left\{ u \;\middle|\; \begin{array}{l} u(t) = (p, w(t))\,, \text{and } u(t) \text{ is a vector-valued piecewise continuous} \\ \text{function on } [0,T], 0 \leqslant t \leqslant T, \text{ such that the solution of (2.5) with} \\ \text{respect to } x(t) \text{ is uniquely determined and all integrals to be} \\ \text{encountered in the sequel exist; moreover, } p > 0, w(t) \geqslant 0 \end{array} \right\}.$$

Let $\tilde{u}(t) = (\tilde{p}_T, \tilde{w}(t))$, denote the optimal strategy, and let $\tilde{\pi}_1(T) := \pi_1(T, \tilde{u}((\cdot))$ denote

[21] See, for example, Transchel and Minner (2009) for a model that considers a discrete number of prices changes larger than one in an inventory control environment.

the maximized profit[22], i.e.,

$$\tilde{\pi}_1(T) = \sup_{u \in \tilde{U}_T} \pi_1(T, u) = \sup_{u \in \tilde{U}_T} \int_0^T \nu(t, u) dt. \tag{2.47}$$

In Section 2.2, we considered dynamic pricing schemes, and the optimal price guaranteed a positive price contribution; recall, $p^*(t) = \bar{p}(t) = \varepsilon(t)/(\varepsilon(t) - 1)c(t) > c(t) \geqslant c(0)$, $0 \leqslant t \leqslant T$. Furthermore, according to the Dorfman-Steiner relation, it is optimal to advertise at a positive rate. If the retailer sets a fixed price $\tilde{p}_T$ for a sales period of length T, three cases need to be considered: (i) $\tilde{p}_T < c_0$, (ii) $c_0 \leqslant \tilde{p}_T \leqslant c(T)$, and (iii) $\tilde{p}_T > c(T)$. In case (i), since the cost function $c(t)$ is nondecreasing, the price contribution $\tilde{p}_T - c(t)$ is negative for every t. Hence, to prevent losses, the retailer will set the (optimal) advertising rate equal to zero and the retailer's total net profit is zero. In case (iii), the price contribution is always positive, and it is profitable for the retailer to set the advertising rate in such a way that the net profit rate is positive. We will show that such dynamic advertising policies exist.

Case (ii), when the (optimal) price lies between $c(0) = c_0$ and $c(T)$, is the most interesting scenario; recall, except for the very special case $\ell(t) = q(t) = r(t) \equiv 0$ and $c(t) \equiv c_0$, the cost function strictly increases. As long as the static price lies above the cost rate, the price contribution is positive and the retailer has an incentive to advertise at a positive rate. This situation is similar to case (iii). When the value of the cost function reaches the price $\tilde{p}_T$, the price contribution is zero, and then becomes negative. Therefore, a rational retailer will set the advertising rate equal to zero to avoid losses; this situation is similar to case (i). To identify an optimal control we need to take a closer look at cases (ii) and (iii).

Let $\tilde{T}(p)$ denote the first time point when the cost function $c(t)$ equals the price value p. Except for the special case where $c(t)$ is (piecewise) constant, the cost function strictly increases in t, and hence, this point $\tilde{T}(p)$ is unique in case (ii). Moreover, the inverse function c^{-1} of the cost function exists. If $p < c_0$, case (i), we set $\tilde{T}(p) \equiv 0$, and if $p > c(T)$, case (iii), we set $\tilde{T}(p) \equiv T$. Thus,

$$\tilde{T}(p) := \begin{cases} 0, & \text{if } p < c_0 \quad (\text{case } (i)), \\ c^{-1}(p), & \text{if } c_0 \leqslant p \leqslant c(T) \quad (\text{case } (ii)), \\ T, & \text{if } p > c(T) \quad (\text{case } (iii)). \end{cases} \tag{2.48}$$

[22] The existence and finiteness of the optimal value and the existence and uniqueness of the optimal control are proved below.

For instance, without discounting and deterioration, i.e., $q(t) = r(t) \equiv 0$, and if the inventory cost rate is constant, i.e., $\ell(t) \equiv \ell$, then the inverse function - and thus the price *threshold* - is given by $c^{-1}(p) = (p - c_0)/\ell$.

From time $\tilde{T}$ onwards, selling goods causes a deficit contribution to the monopolist's profit. Since the advertising rate can be chosen dynamically and in order to avoid such losses, the monopolist will set $w(t) \equiv 0$ for all $t \geqslant \tilde{T}(p)$ if $\delta > 0$.[23] Thus, the optimal dynamic advertising rate depends on time and on the (fixed) price p. Once an optimal dynamic price-dependent advertising rate has been identified, the optimal fixed price will have to be determined numerically.

Therefore, we consider the following two-stage problem: (a) find a (price-dependent) advertising strategy $\tilde{w}(t,p)$ that maximizes

$$\int_0^{\tilde{T}(p)} \nu\left(t, p, w(t)\right) dt = \int_0^{\tilde{T}(p)} e^{-R(t)} \left[\left(p - c(t)\right) p^{-\varepsilon} \mu(t) w(t)^{\delta(t)} - w(t)^{a(t)}\right] dt \tag{2.49}$$

with respect to $w(t)$; (b) maximize expression (2.49) with respect to p using the optimal advertising strategy $\tilde{w}(t,p)$. We shall denote such an optimal (static) price by $\tilde{p}$.

Even for very special parameter settings, e.g., $r(t) = q(t) \equiv 0$, $\Delta(t) = 1/2, \varepsilon = 2$, we are not able to derive a closed form solution of $\tilde{p}$. Only if $\delta(t) = \Delta(t) \equiv 0$, we are able to find a solution formula of the (now pure pricing) model. Since the advertising rate will be set zero as soon as the profit margin becomes negative, and therefore no more sales take place thereafter, the point $\tilde{T}(p)$ is identical to the point in time when the inventory is depleted, i.e., $\tilde{x}(\tilde{T}(p)) = 0$. In our deterministic setting this point in time will be known in advance. When planning her advertising strategy and when deciding on p, the retailer will take the choice of $\tilde{T}(p)$ into account. Thus, analyzing the semi-static model is more complicated than analyzing the model where the actual sales period does not depend on price.

Before we continue with our analysis of the static-price-dynamic-advertising model we present results concerning the optimal control.

Theorem 2.3.2 *Let* $0 \leqslant \delta(t) < a(t) < \overline{a}$, $\Delta(t) = \delta(t)/a(t)$, *and* $\varepsilon > \underline{\varepsilon} > 1$ *for all* $t \in [0,T]$. *Assume* $\mu(t) > 0$, *and let the demand rate equal* $\lambda(t,p,w) = \mu(t) p^{-\varepsilon} w^{\delta(t)}$. *Let the cost function* $c(t)$ *be given by* (2.8), *and assume the price* $p > 0$ *to be arbitrary but*

[23] If $\delta \equiv 0$ such losses can not be avoided due to the lack of advertising effect. Hence, $\tilde{w}(t) \equiv 0 on [0,T]$ and the problem becomes the pure pricing model with dynamic cost and demand, cf. the fixed-price setting in Rajan et al. (1992). We consider this special case in Proposition 2.3.2.

fixed. Then, the dynamic advertising rate $\tilde{w}(t,p)$ *that maximizes* (2.49) *is given by*

$$\tilde{w}(t,p) = \begin{cases} \left(\Delta(t)\mu(t)\dfrac{p-c(t)}{p^{\varepsilon}}\right)^{\frac{1}{a(t)-\delta(t)}}, & \text{if } p > c(t), \\ 0, & \text{else.} \end{cases} \tag{2.50}$$

The value of the optimal price $\tilde{p}_T$ *that maximizes* $\pi_1(T, p, \tilde{w}(t,p))$ *is a solution to the equation*

$$\frac{\partial \pi_1}{\partial p}\left(T, \tilde{p}_T, \tilde{w}(t, \tilde{p}_T)\right) = 0. \tag{2.51}$$

Proof: Assuming the price $p > 0$ to be arbitrary but fixed, the expression to be maximized with respect to the advertising rate is given by the integral, cf. (2.49),

$$\int_0^T e^{-R(t)}\left[(p-c(t))\, p^{-\varepsilon}\mu(t)w(t)^{\delta(t)} - w(t)^{a(t)}\right] dt = \int_0^T \nu(t,p,w(t))dt. \tag{2.52}$$

Whenever $0 < p < c(t), 0 \leqslant t \leqslant T$, the integrand $\nu(t,p,w(t))$ in (2.52) is negative if $w(t) > 0$. If $w(t) = 0$, then the integrand is zero. Thus, it is optimal to set $\tilde{w}(t,p) = 0$ whenever $p \leqslant c(t)$.[24] Since $c(t)$ is a nondecreasing function it is optimal to set the advertising rate equal to zero until time T. Thus, the upper limit of integration in (2.52) can be replaced by $\tilde{T}(p)$, and we get

$$\int_0^{\tilde{T}(p)} e^{-R(t)}\left[(p-c(t))\, p^{-\varepsilon}\mu(t)w(t)^{\delta(t)} - w(t)^{a(t)}\right] dt. \tag{2.53}$$

To maximize (2.53) with respect to functions $w(t)$, we will exploit the special structure of (2.53) and maximize the integrand $\nu(t,p,w)$ with respect to w for eacht t, cf. Section 2.2. Without loss of generality the integrand is positive, since, by definition, $p > c(t)$ on $\left[0, \tilde{T}(p)\right)$. The integrand $\nu(t,p,w)$ is differentiable in w, $\nu(t,p,0) = 0$, and $\lim_{w\to\infty} \nu(t,p,w) < 0$. Hence, for any $0 \leqslant t \leqslant \tilde{T}(p)$, $\nu(t,p,w)$ attains its maximum with respect to w at a finite value $\tilde{w}(t,p) \geqslant 0$. Whenever $\delta(t) = 0$, the maximum is attained at $\tilde{w}(t,p) = 0$. Whenever $\delta(t) > 0$, the optimal advertising rate $\tilde{w}(t,p)$ at time t must satisfy the first order condition, $0 \leqslant t < \tilde{T}(p)$,

$$\frac{\partial \nu}{\partial w}(t,p,\tilde{w}(t,p)) = e^{-R(t)}\left(\delta(t)\mu(t)\frac{p-c(t)}{p^{\varepsilon}}\tilde{w}(t,p)^{\delta(t)-1} - a(t)\tilde{w}(t,p)^{a(t)-1}\right) \stackrel{!}{=} 0. \tag{2.54}$$

[24] If $p = c(t)$, the (net) revenue part of the integrand is zero and $w(t) > 0$ then implies costs only and hence a negative value of the integrand. Thus, $\tilde{w}(t,p) = 0$ is also optimal if $p = c(t)$.

Solving for $\tilde{w}(\cdot)$ yields (2.50); notice that $\tilde{w}$ is the unique solution of (2.54). Moreover, $\tilde{w}$ is a piecewise continuous function on $[0, \tilde{T}(p)]$ since all terms of the right-hand side of (2.50) are piecewise continuous. Furthermore, the values $\tilde{w}(t,p)$ and $\nu(t,p,\tilde{w}(t,p))$ are positive for any pair (t,p) that satisfies $p > c(t)$ and $\delta(t) > 0$, $0 < t < T$. Hence, $\tilde{w}(t,p)$ is the pointwise maximizer of the integrand in (2.53).

We evaluate $\pi_1\left(\tilde{T}(p), p, \tilde{w}(t,p)\right)$ and obtain

$$\begin{aligned}\pi_1\left(\tilde{T}(p), p, \tilde{w}(t,p)\right) &= \int_0^{\tilde{T}(p)} \nu(t,p,\tilde{w}(t,p))dt \\ &\overset{\Delta(t)>0}{=} \int_0^{\tilde{T}(p)} e^{-R(t)} \frac{1-\Delta(t)}{\Delta(t)} \left(\Delta(t)\mu(t)\frac{p-c(t)}{p^{\varepsilon}}\right)^{\frac{1}{1-\Delta(t)}} dt. \qquad (2.55)\end{aligned}$$

Expression (2.55) needs to be maximized with respect to $p > 0$. By definition, $p > c(t)$ on $[0, \tilde{T}(p))$, and the integrand of (2.55) will be nonnegative; thus, $\pi_1\left(\tilde{T}(p), p, \tilde{w}(t,p)\right) \geqslant 0$. If $p = c_0$, then $\tilde{T}(c_0) = 0$ and $\pi_1\left(0, c_0, \tilde{w}(t,c_0)\right) = 0$. If $p = c(T)$, then $\tilde{T}(c(T)) = T$, and the positive profit margin on $[0,T)$ ensures that $\pi_1\left(T, c(T), \tilde{w}(t, c(T))\right)$ is a strictly positive function. If the price is chosen to be *very large*, the total profit becomes small[25]; in particular, $\lim_{p\to\infty} \pi_1(T, p, \tilde{w}(t,p)) = 0$. Notice that by assumption $\varepsilon > \underline{\varepsilon} > 1$, and that the expression $(p - c(t))/p^{\varepsilon}$ converges to zero. Since (2.55) is differentiable in p the function $\pi_1\left(\tilde{T}(p), p, \tilde{w}(t,p)\right)$ attains its maximum at some finite value $\tilde{p}_T > 0$. At $\tilde{p}_T$ the first order condition (2.51) is satisfied.

Since $\tilde{p}_T$ is positive, and $\tilde{w}(t, \tilde{p}_T)$ is piecewise continuous, the policy we have constructed belongs to $\tilde{U}_T$ and is optimal. ♦

In general, we are not able to prove that $\tilde{p}_T$ is unique. In the sequel, we shall assume that $\tilde{p}_T$ is the unique solution of (2.51). The following proposition specifies special cases when the optimal price can be described very precisely.

Proposition 2.3.2 *Assume the conditions that underlie Theorem 2.3.2 are satisfied.*

(a) If $\delta(t) \equiv 0$ for all $t \in [0,T]$, then

$$\tilde{p}_T = \frac{\varepsilon}{\varepsilon - 1} \frac{\int_0^T e^{-R(t)}\mu(t)c(t)dt}{\int_0^T e^{-R(t)}\mu(t)dt}. \qquad (2.56)$$

[25]Below we will show that the profit function decreases in p if $p > \varepsilon/(\varepsilon-1)c(T)$.

(*b*) *If* $\delta(t) > 0$ *for all* $t \in [0, T]$, *then* $\tilde{p}_T$ *satisfies the following inequalities:*

$$\frac{\varepsilon}{\varepsilon - 1} c_0 \leqslant \tilde{p}_T \leqslant \frac{\varepsilon}{\varepsilon - 1} c(T). \tag{2.57}$$

(*c*) *Let the assumptions of* (*a*) *or* (*b*) *be satisfied. If, in addition,* $c(t) \equiv c_0$ *for all* $t \in [0, T]$, *then*

$$\tilde{p}_T = \frac{\varepsilon}{\varepsilon - 1} c_0. \tag{2.58}$$

Proof: (*a*) If $\delta(t) \equiv 0$ on $[0, T]$, then $\Delta(t) \equiv 0$ on $[0, T]$. Hence, by (2.50), $\tilde{w}(t, p) = 0$ for all pairs (t, p), $0 \leqslant t \leqslant T, p > 0$, follows. Thus, the objective function to be maximized with respect to $p > 0$ is

$$\pi_1 (T, p, 0) = \int_0^T e^{-R(t)} \mu(t)(p - c(t)) p^{-\varepsilon} dt. \tag{2.59}$$

Note, since advertising has no effect the limit of integration is T. Equation (2.59) specifies a differentiable function in p. Since $c(t)$ is increasing, except for the special case $c(t) \equiv c_0$, the value of (2.59) is negative if $p < c(0) = c_0$. If $p = c(T)$, then $\tilde{T}(p) = T$ and the integrand of (2.59) is positive, and $\pi_1 (T, p, c(T))$ is positive as well. By Lebesgue's Theorem we may interchange integration and the limit if p tends to infinity. For each t, the integrand goes to zero if p goes to infinity. Thus, $\lim_{p \to \infty} \pi_1 (T, p, 0) = 0$ and $\pi_1 (T, p, 0)$ attains its maximum at some finite value $\tilde{p}_T > c_0$. The optimal price $\tilde{p}_T$ satisfies the first order condition

$$\frac{d \left(\int_0^T \nu(t, p, 0) dt \right)}{dp} (T, \tilde{p}_T, 0) = \int_0^T e^{-R(t)} \mu(t) \frac{(\tilde{p}_T^{\varepsilon} - \varepsilon(\tilde{p}_T - c(t)) \tilde{p}_T^{\varepsilon - 1}}{\tilde{p}_T^{2\varepsilon}} dt \stackrel{!}{=} 0. \tag{2.60}$$

Solving (2.60) for $\tilde{p}_T$ we obtain (2.56). Since $\tilde{p}_T$ is the unique solution of (2.60), $\tilde{p}_T$ is optimal.

(*b*) Since, by assumption, $\delta(t) > 0$ for all $t \in [0, T]$, the objective function to be maximized with respect to p equals

$$\begin{aligned}
\pi_1\left(\tilde{T}(p), p, \tilde{w}(t,p)\right) &= \int_0^{\tilde{T}(p)} \nu(t,p,\tilde{w}(t,p))dt \\
&= \int_0^{\tilde{T}(p)} e^{-R(t)} \frac{1-\Delta(t)}{\Delta(t)} \left(\Delta(t)\mu(t)\frac{p-c(t)}{p^{\varepsilon}}\right)^{\frac{1}{1-\Delta(t)}} dt. \qquad (2.61)
\end{aligned}$$

By Leibniz's rule, the derivative of $\pi_1(\tilde{T}(p), p, \tilde{w}(\cdot))$ with respect to p is given by, $p > c_0$,

$$\frac{\partial \pi_1}{\partial p}\left(\tilde{T}(p), p, \tilde{w}(\cdot)\right) = \int_0^{\tilde{T}(p)} \frac{\partial \nu}{\partial p}\left(t, p, \tilde{w}(t,p)\right) dt + \nu\left(\tilde{T}(p), p, \tilde{w}(\tilde{T}(p), p)\right) \tilde{T}'(p). \qquad (2.62)$$

We will show that the second addend on the right-hand side of (2.62) equals zero.

If $p \in (c_0, c(T))$, case (ii), then, by definition, $\tilde{T}(p) < T$ and $\tilde{w}(\tilde{T}(p), p) = 0$. Thus, $\nu(\tilde{T}(p), p, \tilde{w}(\tilde{T}(p), p)) = 0$. If $p \geqslant c(T)$, case (iii), then $\tilde{T}(p) = T$ and $\tilde{T}'(p) = 0$. Hence, $\nu(\tilde{T}(p), p, \tilde{w}(\tilde{T}(p), p))\tilde{T}'(p)$ equals zero for the cases (ii) and (iii). Elementary calculus shows that, see (2.55) for the formula of $\nu(t, p, \tilde{w}(t,p))$,

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p}\left(\tilde{T}(p), p, \tilde{w}(\cdot)\right) &= \int_0^{\tilde{T}(p)} \frac{\nu\left(t, p, \tilde{w}(t,p)\right)}{1-\Delta(t)} \left(\frac{1}{p-c(t)} - \frac{\varepsilon}{p}\right) dt \\
&= \int_0^{\tilde{T}(p)} \frac{e^{-R(t)}\mu(t)}{p^{\varepsilon}} \left(\Delta(t)\mu(t)\frac{p-c(t)}{p^{\varepsilon}}\right)^{\frac{\Delta(t)}{1-\Delta(t)}} \left(1 - \varepsilon\frac{p-c(t)}{p}\right) dt. \qquad (2.63)
\end{aligned}$$

The first order condition for $\tilde{p}_T$ being optimal implies that (2.63) equals zero if p is replaced by $\tilde{p}_T$. To see that the inequalities (2.57) hold, let us assume that $\tilde{p}_T < \frac{\varepsilon}{\varepsilon-1}c_0$. Since $c(t)$ is nondecreasing, this inequality implies for any $t \in \left[0, \tilde{T}(p)\right)$,

$$\tilde{p}_T < \frac{\varepsilon}{\varepsilon-1}c_0 \quad \Rightarrow \quad \tilde{p}_T < \frac{\varepsilon}{\varepsilon-1}c(t) \quad \Leftrightarrow \quad 1 - \varepsilon\frac{\tilde{p}_T - c(t)}{\tilde{p}_T} > 0.$$

Hence, the right-hand side of (2.63) would be positive, which contradicts the necessary optimality condition for $\tilde{p}_T$; thus, $\tilde{p}_T \geqslant \frac{\varepsilon}{\varepsilon-1}c_0$. Along the same lines one verifies that $\tilde{p}_T \leqslant \frac{\varepsilon}{\varepsilon-1}c(T)$.

(*c*) If $\delta(t) \equiv 0$ and $c(t) \equiv c_0$ on $[0, T]$, then (2.58) follows by evaluating (2.56). If $\delta(t) > 0$ and $c(t) \equiv c_0$ on $[0, T]$, the two inequalities (2.57) become equalities, and (2.58) follows. ♦

In case of a static (time-homogeneous) monopoly environment without inventory cost, i.e., all parameter values are constant over time and no discounting or deterioration effects need to be taken into account, the cost function and the optimal prices are constant over time; more precisely, $\tilde{p}_T = \bar{p}(t) = p^*(t) \equiv \varepsilon/(\varepsilon - 1)c_0$ for all t in $[0, T]$. Consequently, the advertising rate is also constant. We will elaborate on this (classical) results in the following section. In that section, we also compare different static and dynamic settings and discuss numerical and illustrative examples. Moreover, we offer comparative statics results.

2.4 Sensitivity Analysis

In this section, we will further analyze the models which we have studied so far and we will concentrate on how these models are used in practice. In Subsection 2.4.1, we first study how the parameter values, μ, δ, etc., affect the optimal control and the associated optimal sales rates and optimal profit rates. In this context, for example, the following questions are of interest: how is the optimal advertising control affected by a change in the price elasticity ε? How will the optimal profit rate be influenced by a change in ε? How will the initial inventory level change if ε is changing? Since we derived closed-form solution expressions of the optimal dynamic price and dynamic advertising rate, cf. Theorem 2.2.1, we generally draw our conclusions by assuming the parameter of interest to be independent of time and take the first derivative with respect to that value. In the following, we assume all derivatives to exist. Let $el_{x,y}$ denote the elasticity of x with respect to changes of y,

$$el_{x,y} := \frac{\partial x}{\partial y} \frac{y}{x}. \tag{2.64}$$

For instance, $el_{p^*(t),\varepsilon}$ denotes the elasticity of the optimal dynamic price at time t with respect to a change in the price elasticity. We discuss structural results to be derived for the dynamic model, i.e., the model where prices and adverting rates are allowed to be chosen dynamically, cf. Section 2.2, and in Subsection 2.4.1 we present the results of numerical studies which illustrate our findings.

In Subsection 2.4.2, we compare the performance of the dynamic model with the performance of the (partially) static settings. We are particularly interested in how optimal profit margins, optimal capacities, and the associated sales rates depend on the fact that optimal advertising is allowed to be dynamic, instead of being fixed throughout the cycle.

In the following, we will choose several *basic* parameter settings in order to illustrate our general results. Since all quantities of interest depend on the cost function $c(t)$ we will frequently assume a time-homogeneous setting without discounting and deterioration effects, i.e., $q(t) = r(t) \equiv 0$ and $\ell(t) \equiv \ell$; as a consequence, $c(t) = c_0 + \ell t$, cf. Figure 2.1. This simple setting guarantees that the most important characteristic of the cost function - to be an increasing function in t - is assured while the analysis is simplified. In particular, we shall analyze how changes of the price elasticity or the advertising elasticity affect the quantities of interest.

Our analysis is sometimes rather technical but the most important findings related to the optimal dynamic pricing and the optimal dynamic advertising control can be succinctly summarized as follows.

Remark 2.4.1 (management recommendations) *The formulas of the optimal dynamic price p^* and the optimal dynamic advertising rate w^*, cf. Theorem 2.2.1, support the following recommendations:*

- *Optimal dynamic prices are neither influenced by the opportunity to advertise nor by the efficiency of advertising. If the sales rate is boosted by advertising, more customers will purchase the product for the same price.*

- *The optimal dynamic prices are not (directly) influenced by the arrival intensity $\mu(t)$, i.e., optimal prices are independent of the market size or the standard consumer's interest.*

- *The optimal dynamic prices only will be decreasing over time (a market skimming strategy) if the price elasticity increases over time; typically, the optimal prices will increase over time (a market penetration strategy).*

- *If all parameters of the model are constant and the cost function $c(t)$ increases over time, then it is optimal to decrease the expenditures on advertising over time.*

- *The presence of production/purchasing and inventory costs leads to higher prices and lower sales rate. The interest rate and the deterioration rate act like additional cost factors and amplify the effect of higher prices and lower sales rates.*

- *Whenever $\Delta(t) \equiv 0.5$ on $[0, T]$, i.e., $\delta(t) \equiv a(t)/2$, the optimal profit rate equals the optimal expenditures on advertising at time t. If $\Delta(t) < 0.5$, the optimal profit margin is a fraction of the optimal advertising spending. If $\Delta(t) > 0.5$, the optimal profit margin is a multiple of the optimal advertising expenses.*

- *If the optimal advertising rate $w^*(t)$ is larger than one, then a monopolist benefits from an increase in Δ. Both, the revenue rate and the profit rate will increase.*

- *A larger arrival intensity μ increases the optimal profit margin.*

- *The optimal advertising level depends (among other values) on the specific value of the parameters δ and a. The optimal sales rates and the optimal profit rates only depend on the advertising efficiency $\Delta = \delta/a$, but not on the particular values of δ and a.*

2.4.1 Sensitivity Analysis of the Optimal Dynamic Model

In this subsection, we will analyze the optimal dynamic control and associated values with respect to changes in the parameter values. Moreover, we are interested in the behavior of the optimal price, the optimal advertising rate, the associated sales rate, and the optimized profit rate over time. To do so, we analyze the formulas of Theorem 2.2.1. Table 2.1 summarizes our findings; below, we discuss more details and compute elasticities of the quantities of interest with respect to changes of the parameter value, cf. (2.64). The symbol "+" ("−") indicates that the quantity in the column header increases (decreases) if the value of the quantity in the corresponding row header increases. The symbol '·' indicates that the quantity in the column header is unaffected by a change of the quantity in the corresponding row. For example, the optimal dynamic price p^* decreases in the price elasticity ε and increases in the cost function $c(t)$; any change of the advertising efficiency value Δ has no effect on the optimal price. Since the cost function increases in c_0, ℓ, q, and r, see below, we only state $c(t)$ as a row header and implications regarding changes of the components of $c(t)$ are easily deduced. While the results are more or less clear when the value of the arrival intensity or the cost function changes, the implications of a change in ε and in Δ depend on the level of the optimal prices and the optimal advertising rates. For instance, the optimal advertising rate increases in ε if the optimal price is smaller than one and decreases in ε if $p^* > 1$. In the following paragraphs we derive the results of Table 2.1 and discuss these results in more detail.

[26] This only holds if the change in Δ is subject to a change in δ and a is fixed. If δ is fixed and a changes w^* must be larger, respectively smaller, than $e^{-(1/a)}$.

	p^*	w^*	λ^*	ν^*
ε	$-$	$+$ if $p^* < 1$ $-$ if $p^* > 1$	$+$ if $p^* < e^{-\frac{1-\Delta}{\varepsilon-1}}$ $-$ if $p^* > e^{-\frac{1-\Delta}{\varepsilon-1}}$	$+$ if $p^* < 1$ $-$ if $p^* > 1$
Δ	$\cdot$	$+$[26] if $w^* > e^{-(1/\delta)}$ $-$ if $w^* < e^{-(1/\delta)}$	$+$ if $w^* > e^{-(1/a)}$ $-$ if $w^* < e^{-(1/a)}$	$+$ if $w^* > 1$ $-$ if $w^* < 1$
μ	$\cdot$	$+$	$+$	$+$
$c(t)$	$+$	$-$	$-$	$-$

Table 2.1: Summary of the sensitivity results in the dynamic model. A "+" ("$-$") indicates that the quantity in the column header increases (decreases) if the value of the quantity in the corresponding row header increases (at time t). The "$\cdot$" indicates that a change of the value in the row header has no impact on the value in the column header.

The behavior over time

The optimal price policy p^* will typically increase over the sales period. To be precise, the marginal change of p^* at time t, $0 \leqslant t \leqslant T$, is given by

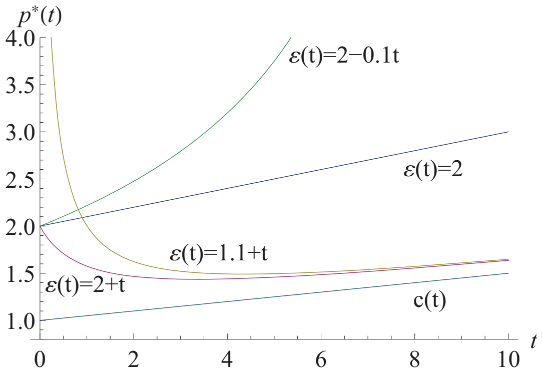
$$\dot{p}^*(t) = p^*(t)\left(\frac{\dot{c}(t)}{c(t)} - \frac{\dot{\varepsilon}(t)}{\varepsilon(t)(\varepsilon(t)-1)}\right); \tag{2.65}$$

alternatively, (2.65) can be formulated in terms of elasticities:

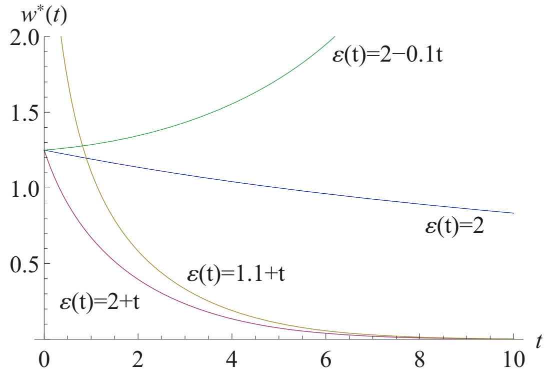
$$el_{p^*(t),t} = el_{c(t),t} - \frac{el_{\varepsilon(t),t}}{\varepsilon(t)-1}. \tag{2.66}$$

Note, $\dot{c}(t)$ is positive except for the special case when $c(t)$ is piecewise constant, cf. equation (2.9). Thus, if the price elasticity is constant, $\varepsilon(t) \equiv \varepsilon$, the subtrahend on the right-hand side of (2.65) equals zero and the relative price increase over time is proportional to the relative cost increase. If the price elasticity varies over time, a decreasing $\varepsilon(t)$ value even boosts this effect, and it is optimal to run a market *skimming* strategy, i.e., prices decrease monotonically over time. Only if customers become more price sensitive, i.e., ε is increasing, then prices are increasing and a market *penetration* strategy is optimal - provided the increase in ε is sufficiently large to compensate for the (relative) increase in the cost function. A similar analysis applies to (2.66), where $el_{c(t),t}$ is strictly positive except when $c(t)$ is piecewise constant. Thus, it depends on the elasticity $el_{\varepsilon(t),t}$ if $el_{p*(t),t}$ is positive or negative.

The left panel of Figure 2.2 depicts optimal price processes considering a linear (in time) elasticity function of the form $\varepsilon(t) = \varepsilon_0 + \varepsilon_1 t > \underline{\varepsilon}, 0 \leqslant t \leqslant T$, and the cost function $c(t) = c_0 + \ell t$ for different choices of ε_0 and ε_1. The optimal prices at time t are bounded from below by the (increasing) cost function $c(t)$. Hence, even if p^* decreases temporarily, the optimal prices cannot decrease *forever* when T becomes large, cf. the case where $\varepsilon(t) = 1.5 + t$. As already pointed out above, we consider an increasing or constant price elasticity as the *typical* case. Especially by assuming ε to increase over time one can model business situations such as retailing out-of-date fashion, perishable food, or outdated movies shortly before they become available for home entertainment. By lowering prices the monopolist counters the lessened willingness-to-pay modeled by an increasing price elasticity. A decreasing price elasticity is less common. A well known example is the sales of airline tickets to business travelers: the closer the date of departure the less important is the ticket fare if it is inevitable to take a certain flight. More generally, such behavior is often observed when a unique or special *event* is close, e.g., the only concert in town, a sports event, or goods whose production will soon be stopped and that become rarities. Other examples for an increasing price elasticity over time are luxury or cultural goods that become more valuable as time passes, e.g., (quality) wine, classic cars, or art. These are not the *classical* goods considered in revenue management but reflect market situations where unique goods lead to a natural monopoly: there is only one agency selling tickets for a certain event, or only one distillery is offering a particular 15 year old scotch whisky. The price set by a company is (usually) observable at the market. Although we face a deterministic demand rate, one can think of the actual price to influence the (potential) customer's buying decision, whereas the advertising rate controls the flow of potential customers. Moreover, the advertising effort of a company is not easily observable or measurable at all. While the monopolist knows the rate at which she runs promotions, the market only observes the effects, e.g., additional or fewer commercials on TV or radio, ads in a magazine, a fluctuating number of sales staff, or varying business hours. We already argued in the remarks following Theorem 2.2.1 that the optimal advertising rate w^* depends on all model parameters (functions). According to equation (2.16), the optimal advertising rate at time t is given by $w^*(t) = c_w(t) \left(\mu(t)/c(t)^{\varepsilon(t)-1}\right)^{1/(a(t)-\delta(t)}$, where c_w is a function of the price elasticity and the advertising elasticity. Assuming constant values for these elasticities ($\Delta(t) \equiv \Delta$ and $\varepsilon(t) \equiv \varepsilon$), the value $c_w(t) \equiv c_w$ is also constant. Then, the dynamics in the optimal advertising process is solely determined by the ratio of the functions $\mu(t)$ and $c(t)$; the



(a)



(b)

Figure 2.2: (a) the optimal price $p^*(t) = \varepsilon(t)/(\varepsilon(t)-1)c(t)$ if $c(t) = 1 + 0.05t$ and for different $\varepsilon(t)$ functions (see labels). (b) the optimal advertising rate $w^*(t)$ according to (2.16), where $\mu(t) \equiv 10$, $a(t) = 2$, $\delta(t) = 1$, $c(t) = 1 + 0.05t$, $q(t) = r(t) = 0$, and different $\varepsilon(t)$ functions.

latter is taken to the power of $\varepsilon - 1$.[27] If the arrival intensity $\mu(t) \equiv \mu$ is constant, it is optimal to decrease the advertising rate over time. Only if more and more customers are arriving, i.e., $\mu(t)$ is an increasing function, it will be profitable to attract even more customers with an increasing advertising rate, see below.[28] Panel (b) of Figure 2.2 depicts the optimal advertising rate for the ε functions considered for the optimal price trajectories (the colors coincide in both panels). Except for the case when the price elasticity decreases over time (the green line) all graphs show the *typical* declining behavior; recall, the arrival intensity is constant. If the price elasticity is decreasing ($\varepsilon(t) = 2 - 0.1 * t, 0 \leqslant t < 10$), it is profitable to *counter* the positive impact on demand by increasing prices, cf. the green line in 2.2a, and *boost* by additional spending on advertising. As already pointed out in the discussion related to (2.22) the level of advertising is of special importance: an advertising rate larger than one implies a boost of demand while a rate below one acts as a demand-damper. The evolution of the optimally controlled demand rate λ^* is best analyzed by making use of the dynamic *Dorfman-Steiner* relation (2.14), which we rewrite as

$$\lambda^*(t) = \frac{\varepsilon(t)}{\Delta(t)} \frac{w^*(t)^{a(t)}}{p^*(t)}, \quad 0 \leqslant t \leqslant T.$$

[27] To simplify the analysis one can choose $\varepsilon = 2$; this particular choice will often lead to *handy* expressions.

[28] In the absence of inventory cost and interest and deteriorating effects, i.e., $c(t) \equiv c_0, \ell(t) = q(t) = r(t) \equiv 0$ for all $t \in [0, T]$, the optimal advertising rate (and the optimal price) is constant for the sales period.

For a *typical* (time-homogeneous) case, so that the optimal price function is increasing and the optimal advertising rate is decreasing, the demand rate decreases over time and the sales rate peaks at the beginning of the sales period. An intuitive explanation of this behavior of λ^* is the following one: goods are purchased at the beginning of the sales period and the cost $c_0 x_0$ has to be paid at time zero. Storing these goods (in order to sell them later) incurs inventory cost, discounting and losses due to deterioration. To compensate for these costs the monopolist increases the prices over time, an action which naturally curbs demand. Since $\Delta(t) < 1$ for all $t \in [0, T]$, there is no incentive to (fully) balance the loss in demand due to higher prices by a measured increase of the advertising rate; and the rate of sales decreases. Certain market (parameter) conditions are needed to create an incentive to increase demand, viz. an increasing arrival intensity $\mu(t)$. If the retailer expects more customers to enter the market, it is profitable to *support* this effect by additional advertising spending. There are other situations when λ^* (temporarily) increases. In the subsequent paragraphs we examine the dependence of the optimal controls on the parameter functions - the *market conditions* - in detail. The behavior of λ^* (and other values depending on the optimal control) can then be deduced in particular cases.

There are many reasons why it is important to study the optimal sales rate. For instance, the optimal rate of sales (together with depreciation effects) determines the initial inventory value, cf. (2.7). Due to this relation the behavior of the optimal inventory capacity with respect to changes in the parameter values can be analyzed using the optimal demand rate. Other quantities of interest are the optimal profit margin and the associated one-cycle profit - definitely the most important values for a retailer. We will often make use of the fact that the profit margin is proportional to the expenditures on advertising at time t, $\nu^*(t) = e^{-R(t)} \frac{1-\Delta(t)}{\Delta(t)} w^*(t)^{a(t)}$, see (2.18); the *co-movement* of ν^* and w^* will simplify the analysis. In particular, we reiterate that, *typically*, the profit margin declines over a cycle if $w^*(t)^{a(t)}$ does; this effect is bolstered by the discounting factor $e^{-R(t)}$. The influence of Δ on both quantities, the optimal advertising rate and the optimal profit margin, will be examined below.

The influence of the cost function

All quantities of interest depend on the cost function $c(t)$ which, in turn, only depends on the inventory parameters c_0, ℓ, and q and on the interest rate r.[29] The value of the cost function at time t clearly increases in all its parameters and thus, the optimal price

[29] Recall, wherever $Q(t)$ and $R(t)$ appear in the cost function expression these terms appear as a sum. From the cost aspect deterioration and discounting are equivalent; cf. (2.8) and comments below.

of time t increases in these parameters as well. In particular, for any t, cf. (2.15),

$$el_{p*(t),c_0} = \frac{c_0}{c_0 + \int_0^t e^{-(Q(s)+R(s))}\ell(s)ds}, \tag{2.67}$$

and[30]

$$el_{p*(t),\ell} = \frac{\ell \int_0^t e^{-(Q(s)+R(s))}ds}{c_0 + \ell \int_0^t e^{-(Q(s)+R(s))}ds}. \tag{2.68}$$

Both elasticities are positive. The impact of a change in the values of the production cost c_0 on price is biggest at the beginning of the sales period. The integral expression which shows up in the denominator of the right-hand side of (2.67) equals zero if $t = 0$. Thus, $el_{p*(0),c_0} = 1$. If $\ell(t) \equiv 0$ for all $t \in [0, T]$, then $el_{p*(t),c_0} = 1$ for all $t \in [0, T]$. If ℓ is positive, then $el_{p*(t),c_0}$ decreases and tends to zero in the course of time. Looking at the price elasticity with respect to the inventory cost parameter, we observe the opposite effect: while a change in ℓ inflicts only small price changes at the beginning (none at $t = 0$) the value of $el_{p*(t),\ell}$ increases monotonically towards one. The reaction of the optimal price due to changes in the sum $Q(t) + R(t)$ goes along the same lines, but is not captured by such *simple* expressions as equations (2.67) and (2.68). In Figure 2.1, see Section 2.2, p. 20, particular choices of cost functions are displayed. Since optimal prices are related to cost by a time-dependent markup factor, Figure 2.1 also shows graphs of optimal prices whenever ε is constant.

Optimal prices increase with values of $c(t)$. The optimal advertising rate w^* decreases in $c(t)$, and so do the optimal sales rate (and thus the optimal inventory capacity) as well as the associated profit margin.[31] The exact amount by which these quantities decrease as a function of time depends on the particular parameter setting.

The influence of the price elasticity

Figure 2.2 displays trajectories of optimal controls for different choices of price elasticity functions. The response of the optimal price and the optimal advertising rate to changes in ε is quantified by, $0 \leqslant t \leqslant T$,

$$el_{p*(t),\varepsilon} = -\frac{1}{\varepsilon - 1}, \tag{2.69}$$

[30] When computing the elasticity with respect to changes in the storage cost we set $\ell(t) = \ell$ and differentiate with respect to ℓ.

[31] Note, the cost function shows up in the denominator of these expressions. The cost function enters these expressions as a function with a positive exponent.

and

$$el_{w^*(t),\varepsilon} = -\frac{\varepsilon}{a(t)-\delta(t)} \log\left(p^*(t)\right); \tag{2.70}$$

to obtain (2.69) and (2.70), we set $\varepsilon(t) = \varepsilon$ and differentiate with respect to ε. The effect on the optimal price is solely determined by the level of ε, which is an implication of the constant price elasticity model. Especially in markets with price insensitive customers, i.e., ε is *close* to one, a small change in the value of ε (or its estimate) can have a large effect on the optimal price; this effect is much smaller if ε is *large*. These findings go in line with practical observations, e.g., ticket prices of a flight with plenty of time prior to departure usually do not fluctuate much; typically, (potential) passengers are not committed to a certain flight time or airline. However, shortly before departure opportunities are limited and potential passengers are less price sensitive. As a consequence, prices typically go up the days before departure. Assuming that the medium to long-term demand is primarily influenced by tourists and frequent flyers who are somewhat flexible as far as the departure time of the flight is concerned, this aspect is responsible for relatively large values of ε on individual flights. In contrast to tourists, the *typical* business traveler is known for short-term booking and small price sensitivity. Thus, $\varepsilon(t)$ should be assumed to decreases over time and, as a consequence, prices will increase (especially during the last days before departure). This fact is, basically, not due to the increasing cost function but (mainly) because of the time-inhomogeneous price sensitivity. The interplay of the optimal advertising rate and ε is less intuitive. This interaction actually depends on the current price level. If $p^* > 1$, it is optimal to reduce the advertising rate with increasing ε. This property is reasonable as the markup on the cost rate decreases and promotion becomes (relatively) more expensive. If $p^* < 1$, i.e., $\log(p^*) < 0$, the reaction of the optimal advertising rate with respect to changes in the price elasticity is positive: it is optimal to *fuel* demand, although the markup on prices shrinks. This odd behavior is due to the fact that if the price p is between zero and one, then the term $p^{-\varepsilon} = 1/(p^{\varepsilon})$ increases in ε, i.e., the demand increases, ceteris paribus. Setting $c_0 \equiv 1$, the cost function $c(t)$ will be larger than one. Then, $\varepsilon(t)/(\varepsilon(t)-1)c(t) > 1$ for all $t \geqslant 0$, and it can be guaranteed that the optimal prices are larger than one and w^* is decreasing in ε.[32]

The profit rate associated with the optimal control can be rewritten in terms of the advertising spending, cf. (2.18). The monopolist will only benefit from an increasing price elasticity if optimal prices are below one. The reaction of the optimal sales rate with respect to changes of ε also depends on the current value of Δ. Sales are boosted if

[32] If $c_0 \equiv 1$, one can think of the unit cost c_0 as unit of account. All other costs and prices can be considered in relation to this unit.

the optimal prices are *sufficiently* small, viz. if $p^* < e^{-\frac{1-\Delta}{\varepsilon-1}}$; observe $e^{-\frac{1-\Delta}{\varepsilon-1}} < 1$. These results follow from the evaluation of the corresponding elasticity expressions and can be easily verified by elementary calculus:

$$el_{\lambda^*(t),\varepsilon} = \frac{\varepsilon}{\varepsilon - 1}\left(1 - \frac{\varepsilon - 1}{1 - \Delta(t)} \log\left(p^*(t)\right)\right),$$

and

$$el_{\nu^*(t),\varepsilon} = -\frac{\varepsilon}{1 - \Delta(t)}\left(p^*(t)\right) = a(t)\, el_{w^*(t),\varepsilon}.$$

The influence of the advertising parameters a and δ

In some cases, when analyzing the effect of a change in the advertising elasticity δ and the advertising cost parameter a, we can restrict the analysis to a change of the advertising efficiency parameter Δ. Note, the expression $w^*(t)^a$ only depends on $\Delta = \delta/a$, cf. (2.16). Since the optimal sales rate and the optimal profit rate can be written in terms of Δ and $w^*(t)^a$, cf. (2.17) and (2.18), these quantities of interest can thus be analyzed as functions of Δ. The dependence on a and δ follows by observing that the advertising efficiency parameter Δ increases in δ and decreases in a.

We calculate the elasticity of the advertising spending $w^*(t)^a$ with respect to the advertising efficiency parameter Δ as well as the elasticity of the advertising rate $w^*(t)$ with respect to a and δ[33]:

$$el_{w^*(t)^a,\Delta} = \frac{1 + \delta \log\left(w^*(t)\right)}{1 - \Delta}, \qquad el_{w^*(t),\delta} = \frac{1 + \delta \log\left(w^*(t)\right)}{a - \delta}, \tag{2.71}$$

and

$$el_{w^*(t),a} = -\frac{1 + a \log\left(w^*(t)\right)}{a - \delta}. \tag{2.72}$$

It follows from (2.71) and (2.72) that the optimal advertising rate increases in δ (and Δ) and decreases in a: since advertising becomes more efficient (or less expensive), the promotional effort is boosted. Like in the case of price elasticity, there is an exception to this rule. If the optimal advertising rate is so small that $\delta \log(w^*(t)) < 1$, then $el_{w^*(t)^a,\Delta}$ and $el_{w^*(t),\delta}$ become negative. Panel (a) of Figure 2.3 depicts three fundamental situations: a constant Δ throughout the whole sales horizon (Δ_1 and Δ_2), a first increasing and then decreasing Δ function (Δ_3), and a first decreasing and then increasing (*U-shaped*) Δ function (Δ_4). Panel (b) shows the optimal advertising rate for the different Δ-functions; we set $a(t) \equiv 2$ and only vary the δ values. For simplicity, ε

[33]When computing the elasticity with respect to changes in the parameter of interest, we set $a(t) = a$, $\delta(t) = \delta$, and $\Delta(t) = \Delta$ and differentiate with respect to a, δ, or Δ.

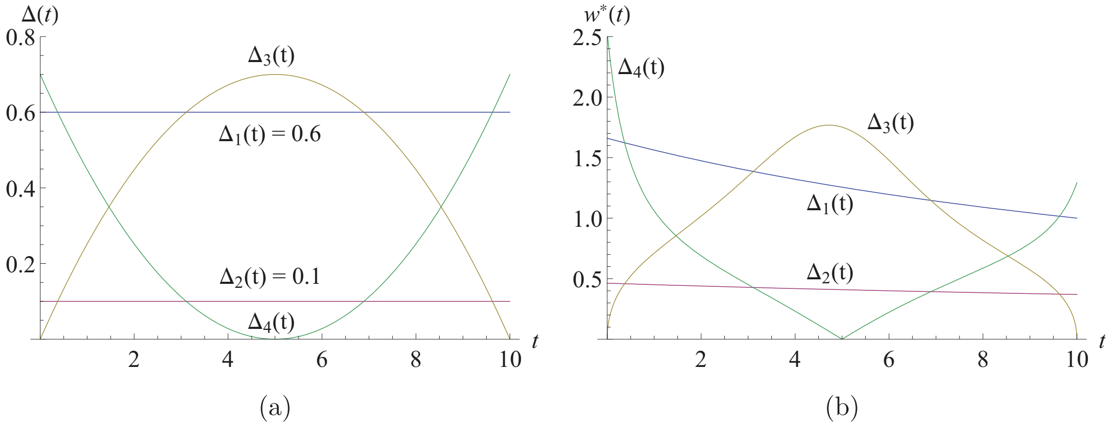


Figure 2.3: (a) different Δ-functions defined on the interval $[0, T]$, $T = 10$: $\Delta_1(t) = 0.6$, $\Delta_2(t) = 0.1$, $\Delta_3(t) = 0.7\frac{t(T-t)}{(T/2)^2}$, $\Delta_4(t) = 0.7 - \Delta_3(t)$. (b) the optimal advertising rate $w^*(t)$ according to (2.16), where $\varepsilon = 2$, $\mu(t) = 10$, $c(t) = 1 + 0.05t$, $q(t) = r(t) = 0$, and different Δ-functions ($a(t) \equiv 2$).

and μ are constants and no discounting nor deterioration is considered ($q(t) = r(t) \equiv 0$). If Δ is constant (Δ_1 and Δ_2), then the optimal advertising rate strictly decreases over time; in the case when Δ is small (Δ_2), the rate is almost constant. When Δ changes over time the optimal advertising rate follows the pattern of $\Delta(t)$, peaking at the same point in time.[34] In contrast to the Δ function, $w^*(t)$ is not symmetric since the cost function increases over time, cf. (2.16). If additionally $c(t) \equiv c_0$, one can indeed expect constant optimal advertising rates when $\Delta(t)$ is constant, and symmetry around $T/2$ for the dynamic cases Δ_3 and Δ_4. The asymmetric pattern for an increasing cost function is persistent when we examine the associated optimal sales rates and profit margins as shown in Figure 2.4. Most interestingly, the profit margin ν^* is smaller for $\Delta_1 = 0.6$ than for $\Delta_2 = 0.1$. Although advertising is more effective and takes a larger value for all $t \in [0, T]$, cf. panel (b) of Figure 2.3, the monopolist reaps lower profit at *all* times. Recall, the profit rate at time t can be expressed in terms of the advertising cost $w^*(t)^a$ times the Δ-dependent factor (and the discounting factor which, for simplicity, is set to one in the case at hand), namely $\nu^*(t) = \frac{1-\Delta(t)}{\Delta(t)} w^*(t)^{a(t)}$, cf. Theorem 2.2.1. Actually, there are two opposing effects at work. First, a larger value of Δ drives up the advertising expenses $w^*(t)^a$. Second, a larger Δ value reduces the value of the quotient $\frac{1-\Delta(t)}{\Delta(t)}$. For our illustrative parameter setting the optimal adverting cost rate is a square function ($a(t) = 2$), i.e., $w^*(t)^2 \approx 1.5^2 = 2.25$ in the case of $\Delta_1 = 0.6$, and $w^*(t)^2 \approx 0.5^2 = 0.25$ in the case of $\Delta_2 = 0.1$. The values of the Δ-dependent factors are $(1 - \Delta_1)/\Delta_1 = 2/3$

[34]This is not true in general as indicated above when $w^*(t)$ takes very small values, cf. (2.71).

and $(1-\Delta_2)/\Delta_2 = 9$, respectively. Overall, for a monopolist it is more favorable to face the advertising efficiency Δ_1 than to face Δ_2.

Although λ^* and ν^* fluctuate over time, the optimal sales rate and the optimal profit rate might drop if w^* is small and if Δ increases, cf. the scenario Δ_3 in Figure 2.4. The corresponding elasticity formulas are given by

$$el_{\lambda*(t),\Delta} = \frac{\Delta}{1-\Delta}\left(1 + \log\left(w^*(t)^a\right)\right), \tag{2.73}$$

and

$$el_{\nu*(t),\Delta} = \frac{\Delta}{1-\Delta}\log\left(w^*(t)^a\right). \tag{2.74}$$

Thus, if $w^*(t) > e^{-1/a}$, then $el_{\lambda*(t),\Delta}$ is positive, i.e., an increase in Δ leads to higher

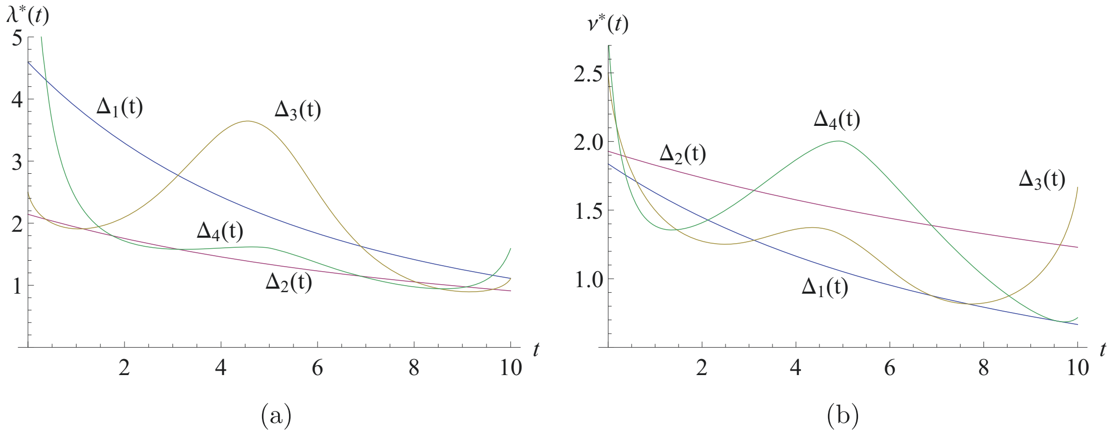


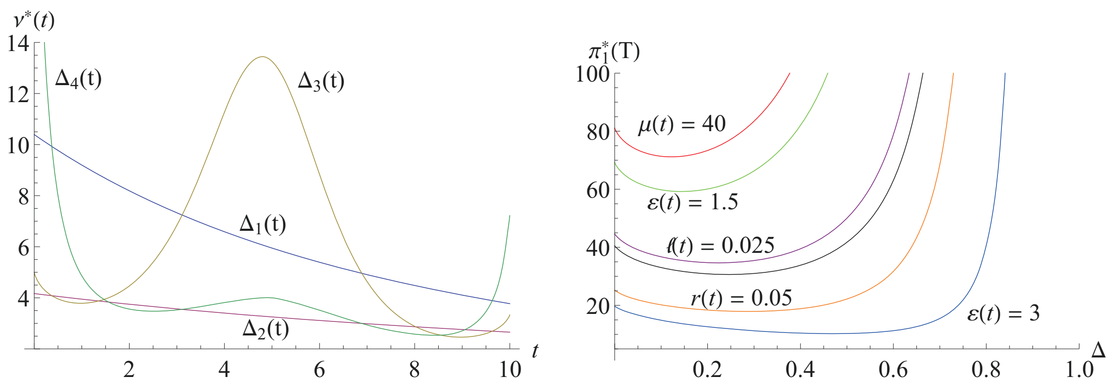
Figure 2.4: (a) the optimally controlled demand rate on $[0, T]$ for different Δ-functions. (b) the optimally controlled profit rate on $[0, T]$ for different Δ-functions. The parameters are: $T = 10$, $\varepsilon = 2$, $\mu(t) = 10$, $c(t) = 1 + 0.05t$, $a(t) = 2$, $q(t) = r(t) = 0$, and the Δ-functions given in panel (a) of Figure 2.3.

sales. In the illustrations, we choose $a \equiv 2$ and the inequality condition becomes $w^*(t) > 1/\sqrt{e} \approx 0.6$. This condition explains the fluctuating behavior of λ^* in the two cases Δ_3 and Δ_4, cf. panel (b) of Figure 2.3 and panel (a) of Figure 2.4.

Similarly, whenever $w^*(t) < 1$, then $\log\left(w^*(t)^a\right) < 0$ and the right-hand side of (2.74) becomes negative, i.e., the optimal profit margin ν^* decreases in Δ. If the market conditions are *favorable* for the monopolist, e.g., a relatively large μ value and/or a small ε value, see above, it is optimal to stimulate the demand by setting $w^*(t) > 1$, and the monopolist always benefits from an increase in Δ, i.e., $el_{\nu*(t),\Delta} > 0$. Whenever it is best to slow down the arrival rate by setting $w^*(t) < 1$, an increasing Δ value diminishes the slow-down factor and dampens the *mitigation* strategy. Recall, advertising at a rate below one curbs the demand. This behavior is counter-intuitive. The common believe is that as the advertising efficiency increases one expects the monopolist to be better off. But the math reveals that this is not the case in the example depicted in Figure 2.3 and Figure 2.4. For various Δ scenarios, the optimal advertising rate falls below one. The condition that $w^*(t)$ must be larger than one to observe the behavior one does expect is equivalent to condition (2.22). In Figure 2.4, we set $\varepsilon(t) \equiv 2$ and condition (2.22) becomes

$$\mu(t) > 4\frac{c(t)}{\Delta(t)} = 2\frac{p^*(t)}{\Delta(t)}. \tag{2.75}$$

If Δ is close to zero, the value $\mu(t) = 10$ - as chosen in Figures 2.3 and 2.4 - is simply not large enough to entice the monopolist to attract more customers. Analyzing the two cases when the advertising efficiency is constant over time ($\Delta_1(t) \equiv 0.6$ and $\Delta_2(t) \equiv 0.1$) shows that the arrival intensity μ must be bigger than (approximately) $10.4c(t)$ to guarantee higher profits under Δ_1 than under Δ_2 at *every* point t.[35] Equivalently, one can think of a given value μ and compute a critical value of the price elasticity such that $w^* > 1$. Panel (a) of Figure 2.5 depicts the profit margin for the four Δ-scenarios when $\mu(t) \equiv 20$,



(a) (b)

Figure 2.5: (a) the optimal profit rate on $[0, T]$ for different Δ-functions, where $\mu(t) \equiv 20$, $\varepsilon = 2$, $c(t) = 1 + 0.05t$, $a(t) = 2$, $q(t) = r(t) = 0$, and the Δ-functions given in Figure 2.3. (b) the optimal one-cycle profit depending on Δ, where Δ is assumed to be fixed throughout the cycle of length T. The black line corresponds to $T = 10$, $\mu(t) \equiv 20$, $\varepsilon = 2$, $c(t) = 1 + 0.05t$, $a(t) = 2$, $q(t) = r(t) = 0$, $\ell(t) = 0.05$; the colored lines deviate by the value labeled at the corresponding graph.

[35] If $r(t) \equiv 0$ and $\varepsilon(t) \equiv 2$, then $\nu^*(t, \Delta) := \nu^*(t) = \frac{1-\Delta}{\Delta}\left(\frac{\Delta}{4}\frac{\mu(t)}{c(t)}\right)^{\frac{1}{1-\Delta}}$. Comparing ν^* for $\Delta_1 = 1/10$ and $\Delta_2 = 3/5$ it is easy to verify that the inequality $\nu^*(t, \Delta_2) > \nu^*(t, \Delta_1)$ is satisfied if (approximately) $\mu(t) > 10.4c(t)$.

ceteris paribus. Now - with a larger μ value - the optimal profit rates co-move with the Δ functions for most of the time; the decrease of ν^* for the Δ_1 and Δ_2 scenario is due to the increasing cost function. Since $\nu^*(t)$ is only a *snapshot* of the total profit per cycle at time t, from a practical point of view, it is more relevant to analyze the optimal total profit per cycle with respect to changes of Δ (and other parameters). Panel (b) of Figure 2.5 illustrates how the one-cycle profit depends on (constant) Δ-values (horizontal axis). The different graphs of $\pi_1^*(T)$ as a function of Δ illustrate variations of the basic parameter setting $\mu(t) \equiv 20$, $\varepsilon = 2$, $c(t) = 1 + 0.05t$, $a(t) = 2$, $q(t) = r(t) = 0$, $\ell(t) = 0.05$. The profit function related to this particular parameter setting is represented by the black line. The label next to a line indicates which basic parameter has been changed and displays the new value. The levels of the profit lines vary, but all lines show a convex behavior. Most interestingly, for all parameter settings one observes a decreasing behavior for small Δ-values, i.e., it would be favorable for the monopolist to operate on a market where advertising has no impact ($\delta(t) = \Delta(t) \equiv 0$), but customers were to arrive by intrinsic motives. However, if Δ is large, advertising is beneficial.

The influence of the arrival intensity μ

The analysis of how quantities of interest depend on the arrival intensity μ - the customer's basic arrival rate - is straightforward: the optimal price does not depend on μ. Thus, neither the values nor changes of μ affect p^*. The optimal advertising rate, the optimal rate of sales and the optimal profit rate co-move with the arrival intensity. The arrival rate μ enters all formulas as a power expression; the exponent only depends on a and δ. Hence, the corresponding elasticities are given by the following *simple* expressions[36]:

$$el_{w^*(t),\mu} = \frac{1}{a(t) - \delta(t)},$$

and

$$el_{\lambda^*(t),\mu} = \frac{1}{1 - \Delta(t)} = el_{\nu^*(t),\mu}.$$

A higher arrival intensity is always profitable for a monopolist. Furthermore, it is best to synchronize the advertising rate with the time-varying arrival rate. The interplay of $\mu(t)$ and $c(t)$, see (2.22), determines whether it is optimal to advertise at a rate above one or at a rate below one. Recall, a rate higher (lower) than one implies boosting (curbing) the rate of sales. We analyze the maximized cycle profit in the next section 2.4.2. There,

[36] When computing the elasticity with respect to changes in the arrival intensity, we set $\mu(t) = \mu$ and differentiate with respect to μ.

we also compare the dynamic model with the partially static models.

The return on sales

Another economic key figure of interest is the return on sales, also referred to as operating margin. The definition of the (total) return on sales is the total profit in relation to total revenue or, in everyday language, how many cents of each dollar taken in line a monopolist's pockets. We define the total return on sales ROS^* associated with the optimal control as

$$ROS^* := \frac{\int\limits_0^T \nu^*(t)dt}{\int\limits_0^T e^{-R(t)} p^*(t)\lambda^*(t)dt} \overset{\substack{\varepsilon(t)\equiv\varepsilon \\ \Delta(t)\equiv\Delta}}{=} \frac{1-\Delta}{\varepsilon}. \tag{2.76}$$

If the price and advertising elasticities are constant over time ($\varepsilon(t) \equiv \varepsilon$ and $\Delta(t) \equiv \Delta$), it is easy to see that $ROS^* = (1-\Delta)/\varepsilon$. We define the *pointwise* return on sales $ros^*(t)$, $0 \leqslant t \leqslant T$, as

$$ros^*(t) := \frac{e^{R(t)}\nu^*(t)}{p^*(t)\lambda^*(t)} = \frac{1-\Delta(t)}{\varepsilon(t)}; \tag{2.77}$$

it is the (adjusted) optimal profit margin[37] divided by the revenue rate at time t. That ros^* is equivalent to $(1-\Delta(t))/\varepsilon(t)$ is evident by recalling the *Dorfman-Steiner* relation (2.14), namely $w^*(t)^{a(t)}/\left(p^*(t)\lambda^*(t)\right) = \Delta(t)/\varepsilon(t)$, and the equation (2.18), which is equivalent to $w^*(t)^{a(t)} = e^{R(t)}\Delta(t)/(1-\Delta(t))\nu^*(t)$. Note, we compute the $ros^*(t)$ at current value at time t whereas the ROS^* is defined in terms of the present value at time zero.

Both terms, the return on sales and the total return on sales, decrease in ε, i.e., the more price sensitive customers are, the smaller is the profit margin relative to the revenue. The optimally controlled return on sales also decreases in Δ, a property which is intuitively clear. The opportunity to advertise admittedly increases the sales rate if $w^* > 1$, but also incurs the cost of *generating* additional demand. Optimal prices are not affected by Δ. Thus, if $w^* > 1$, larger revenue rates and larger profits, cf. the discussion on pp. 56-60, can be observed, but the (pointwise) return on sales will decrease if advertising becomes more efficient.

[37]The profit margin $\nu(t)$ is defined in terms of time zero dollars. Therefore, we multiply the numerator by $e^{R(t)}$ to account for the current value at time t.

2.4.2 Optimal Dynamic vs. Optimal (Partially) Static Models

In the previous section, we focused on the effects of parameter changes on the optimal controls and the optimally controlled system. Now, we compare the dynamic model with the (partially) static models. Instead of analyzing a myriad of different models, we will discuss an exemplary case study and summarize our general findings at the end of this section. The numerical values of the following Example 2.4.1 are motivated by a report on the situation and future of the German (food) discounter market, cf. GfK (2008). We consider a manager who sets the pricing and advertising policy of a supermarket for the sales period of two weeks. Such an example is not a typical inventory model. In particular, it is not realistic to assume that a supermarket stocks the total inventory for the whole planning horizon only once. This would imply that after two weeks all shelves are empty and would all be replenished at one stroke. Instead, we assume that replenishments will occur at several times during the 2-week period: fruits and vegetables twice a week, most other products once a week. We abstract from particular goods or items to be sold and introduce a so-called *shopping card unit* (SCU): an average basket of goods. Furthermore, we assume the price and the advertising rate to be set relative to their basic levels p_0 and w_0, respectively. Then, the factor $\mu(t)$ in the demand rate (2.4) is given by $\mu(t) \equiv (1/w_0)^\delta \, \mu_0(1/p_0)^{-\varepsilon}$, cf. Section 2.2, p. 14. By choosing this representation one can directly interpret the given parameter values as done in the example, see below.

We introduce the relevant data in Example 2.4.1 and discuss these values afterwards. Then, we illustrate the optimal price and advertising policies of our three models: *dynamic-price-dynamic-advertising*, *dynamic-price-static-advertising*, and *static-price-dynamic-advertising*, cf. Figure 2.6. Of particular interest is a comparison between the dynamic-price-dynamic-advertising model - the *benchmark* model - and the static-price-dynamic-advertising model - the most relevant practice model. We compare the profit margins associated with the optimal policies, see Figure 2.7, and the optimal profits of the whole sales period, see Table 2.2. Moreover, we analyze the *inventory* capacities (the initial number of SCUs) and the sales rates associated with the optimal price and advertising policies and we shortly discuss how the sales rates fit to observable customer behavior.

Example 2.4.1 *A supermarket sells a variety of goods, mainly food and convenience goods, from Monday to Saturday. Management wants to set the pricing and advertising policy for (the next) two weeks ($T \equiv 2$). Instead of considering particular items, we concentrate on the sale of an average basket of goods, a so-called shopping cart unit (SCU). Assume that the unit cost of an SCU amounts to 15 dollars ($c_0 \equiv 15$). The inventory* *costs, including attributable labor, insurance, and opportunity costs, add up to one dollar per SCU per week ($\ell(t) \equiv 1$). Customers are assumed to be very price sensitive; if prices are too high, they do their shopping at another supermarket.*[38] *Advertising has only a moderate appeal. The price and advertising elasticities are $\varepsilon(t) \equiv 8$ and $\delta(t) \equiv 0.15$. The advertising cost function is linear ($a(t) \equiv 1$).*

Assuming a weekly advertising effort of $1,000$ *dollars* ($w_0 = 1,000$) *and a price at unit cost* ($p_0 \equiv c_0 = 15$)*, on average,* $\mu_0 = 20,000$ *SCUs will be sold at unit cost per week.*[39]

Typically, customers are sensitive with respect to price changes at a particular store and decide to buy at another store if prices are too high. A value for the price elasticity of $\varepsilon = 8$ is fairly large but not uncommon in food retailing. For example, Hoffmann and Hackelbusch (2013) estimate the value of the (cross) price elasticity of trademark goods between German discounters to be around 13. The choice $p_0 \equiv c_0$ is motivated by the question of how many customers will buy one SCU at cost. Since $a(t) = 1$, the advertising rate can be interpreted as the amount of dollars spent per week. We assume w_0 to correspond to the weekly advertising effort (in dollars) that is *typically* needed to run such a store, i.e., direct promotions (ads, flyers, etc.) and indirect expenditures on advertising (sales staff, *opening hours*, etc.). If the supermarket management decides to set the price $p(t) \equiv c_0 = 15$ and to advertise at rate $w(t) \equiv w_0 = 1,000$, the model postulates that $\mu_0 = 20,000$ customers would be attracted. If the advertising rate is set to $w(t) \equiv 500$, ceteris paribus, this reduces the number of (potential) customers by 10%; note, that the factor $(w(t)/w_0)^\delta$ in the demand rate is then $(500/1000)^{0.15} \approx 0.9$. Due to the small δ-value, even at an advertising rate of only one dollar per week 35 percent of the buyers are still willing to buy an SCU if the price equals p_0. The effect of the choice of w_0 clearly depends on δ, but whenever $\delta(t) > 0$, no customers will show up if no money is spent on advertising. This example will illustrate the importance and the implications of the parameter values and their interpretation. The value of the μ function needed in order to calculate the optimal policies is given by $\mu(t) = (1/w_0)^{0.15}\,\mu_0\,(1/c_0)^{-8} \approx 1.82 \cdot 10^{13}$ - a value that numerically is not always easy to handle. According to our interpretation, μ is the number of SCUs sold at a price of *one* dollar if the advertising rate equals *one*.[40] Alternatively, one can set $\mu(t) \equiv \mu_0$ and one obtains the optimal price and advertising rate in terms of markups on c_0 and w_0, respectively. For example, an optimal price of 1.2

[38] Here, we assume no monopolistic retailer, but one can think of a local monopoly, e.g., the only supermarket in a neighborhood. If the prices are relatively high compared to other stores nearby, people are willing to do the shopping there.

[39] Since the actual sales price will be higher ($\approx$ 18 dollars), the actual number of sales per week will be much lower, see below.

[40] This amount is fairly large but buying a 15 dollar SCU for one dollar only is a really *hot deal.*

then corresponds to a value of $1.2c_0 = 18$ (dollars). Figure 2.6 depicts the optimal price and advertising policies according to the theoretical results presented so far. The optimal price in the *static-price-dynamic-advertising* model is $\tilde{p} \approx 18.25$, which corresponds to a markup of approximately 22 percent relative to the unit cost $c_0 = 15$. The optimal

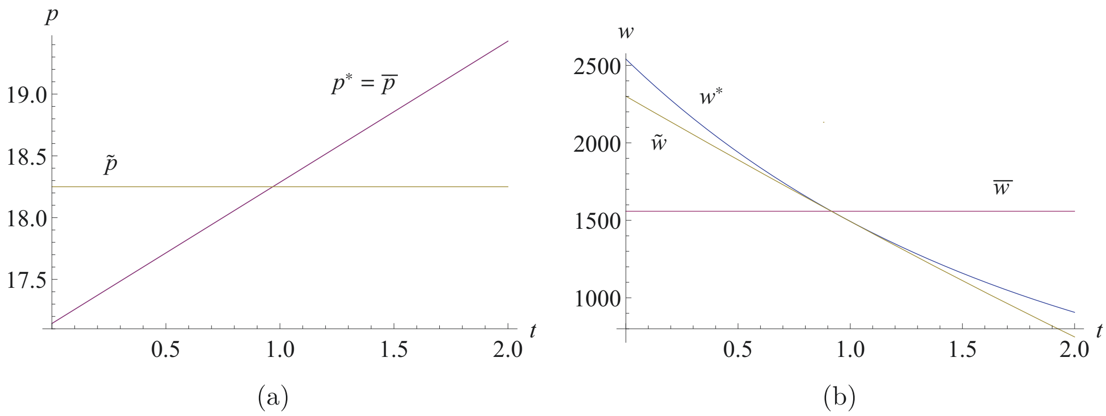


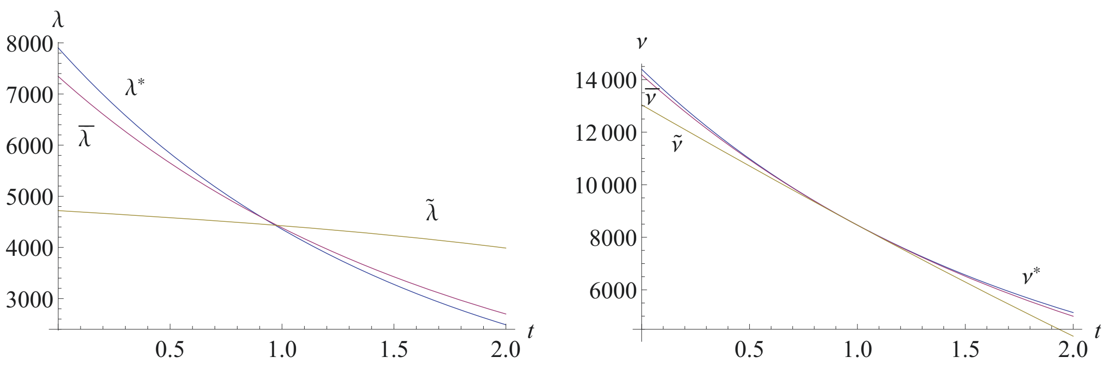
Figure 2.6: Optimal prices (a) and optimal advertising rates (b) in the dynamic and partially fixed control model for Example 2.4.1.

dynamic prices in case of the *dynamic-price-dynamic-advertising* and the *dynamic-price-static-advertising* setting are increasing (in t) and are proportional to the cost function $c(t) = 15+t$.[41] The (theoretical) markup on the unit cost equals $\varepsilon/(\varepsilon-1)-1 = 1/7 \approx 0.14$ at the beginning ($t = 0$) and increases linearly to $\varepsilon/(\varepsilon-1)c(T)/c_0-1 = 31/105 \approx 0.3$ over the two week period; the average optimal dynamic price is approximately equal to the optimal fixed price.[42] In the retail sector a markup between 15 and 30 percent (depending on the goods and the cost structure) is realistic.[43] The different pricing policies have only little influence on the dynamic advertising rates, cf. Figure 2.6b. Although $w^*(t) > \tilde{w}(t)$ at every point t, the values essentially differ only at the beginning and at the end of the considered time horizon ($w^*(0) \approx \$2,540$ compared to $\tilde{w}(0) \approx \$2,300$, and $w^*(T) \approx \$900$ compared to $\tilde{w}(T) \approx \$750$, respectively); note, since $a(t) = 1$, $w^*(t)$ corresponds to the advertising cost rate. In both cases, it is optimal to start with a *high* advertising rate and decrease the promotional effort monotonically over time. When the advertising rate has to be fixed, it is optimal to set the value slightly below the average optimal dynamic rate of the *dynamic-price-dynamic-advertising* setting ($\bar{w} \approx \$1,559 < \$1,569 \approx \int_0^T w^*(t)dt$).

[41] Recall, the optimal dynamic price is the same regardless of whether advertising is dynamic or static.
[42] The average optimal dynamic price evaluates to \$18.2857, whereas the optimal fixed price equals \$18.2548 (precise up to four digits).
[43] See IfH (2009) for estimates of the margin of the German retail industry.

In all three cases advertising is used to boost the sales rate. However, the effect is limited by the small advertising elasticity, e.g., $\bar{w}^{0.15} \approx 3$.

Panel (a) of Figure 2.7 shows the sales rates of the three models considered. While λ^* and $\bar{\lambda}$ more or less co-move, $\tilde{\lambda}$ is almost constant! This has large implications in practice as retailers often seek to smooth their sales over the planning horizon. Here, this effect is achieved by setting a fixed price and advertising dynamically. Customers steadily make their purchases with only a slight decline over time. When the prices increase over time, most sales occur at the beginning of the sales period and the value of the optimal rates λ^* and $\bar{\lambda}$ drops to approximately one third of their initial value close to the end of the sales period. Figuratively speaking, many customers storm the market at the beginning of the period when prices are low before they start increasing. Such a rush implies extra costs which are, to some extent, taken into account by the higher advertising expenses (more sales staff and/or extended opening hours). However, the cost function c only depends on the *internal* cost parameters. It does not capture such external cost effects. In general, the inventory cost $\ell(t)$ can be modeled in such a way to account for some of these effects.



(a) (b)

Figure 2.7: Optimally controlled sales rates (a) and optimal profit margins (b) in the dynamic and partially fixed control model for Example 2.4.1.

In Example 2.4.1, we do not consider deterioration effects, i.e., $q(t) \equiv 0$. Hence, the initial inventory value associated with the optimal control in each of the models considered equals the amount of total sales. The values of the initial inventory of the three models are: $x_0^* \approx 9,260$, $\bar{x}_0 \approx 9,186$, and $\tilde{x}_0 \approx 8,651$. Despite the different model settings, which imply different optimal policies, the differences between the total number of sales of each model are quite small. The differences of the profit margins are also small. Panel (b) of Figure 2.7 depicts the optimally controlled profit margins of the three model variants. It is apparent that all three graphs strictly decrease over time. Hence, profits are mainly gathered in the first half of the sales period[44]; at time T, the values of the profit rates have more than halved compared to their initial values. But the differences between the profit rates, especially between ν^* and $\bar{\nu}$, are small. These observations go in line with the numerical values in Table 2.2. The first row displays

	π_1^*	$\bar{\pi}_1$	$\tilde{\pi}_1$	π_1^{fix}
reference value ($a(t) = 1, \delta(t) = 0.15, \varepsilon = 8$, $\mu_0 = 20,000, q(t) = 0$)	$17,779$	$17,663$ (99.4%)	$17,037$ (95.8%)	$16,922$ (95.2%)
modification to example 2.4.1				
$\varepsilon \equiv 6$	$29,151$	$29,053$ (99.7%)	$28,489$ (97.7%)	$28,392$ (97.4%)
$\varepsilon \equiv 10$	$11,914$	$11,787$ (98,9%)	$11,128$ (93.4%)	$11,005$ (92.4%)
$\delta(t) \equiv 0$	$19,442$	$19,442$ (100%)	$18,746$ (96.4%)	$18,746$ (96.4%)
$\delta(t) \equiv 0.3$	$21,948$	$21,530$ (98.1%)	$20,855$ (95.0%)	$20,438$ (93.1%)
$\mu_0 \equiv 10,000$	$7,866$	$7,815$ (99.3%)	$7,538$ (95.8%)	$7,487$ (95.2%)
$\mu_0 \equiv 30,000$	$28,647$	$28,460$ (99.3%)	$27,451$ (95.8%)	$27,266$ (95.2%)
$q(t) \equiv 0.05$	$13,090$	$12,839$ (98.1%)	11.524 (88.0%)	$11,268$ (86.1%)

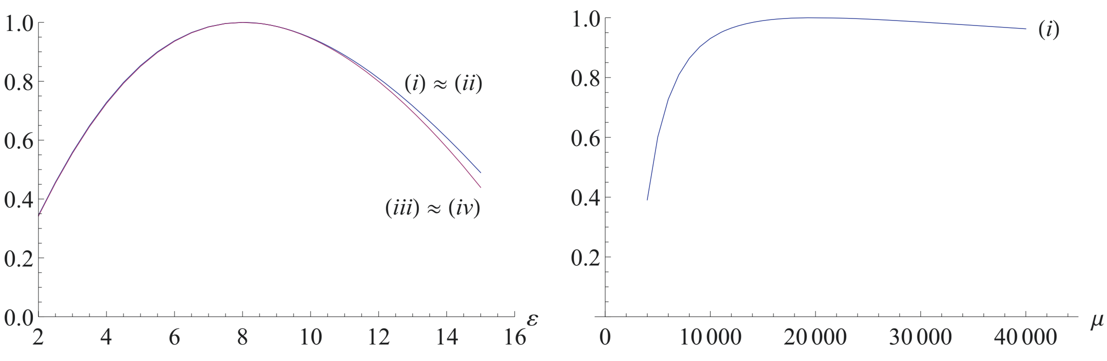
Table 2.2: The optimal profits (in dollars) of four different models and different parameter values: π_1^* - dynamic-pricing-dynamic-advertising, $\bar{\pi}_1$ - dynamic-pricing-static-advertising, $\tilde{\pi}_1$ - static-pricing-dynamic-advertising, π_1^{fix} - static-pricing-dynamic-advertising. The first column indicates changes to the original parameter setting of Example 2.4.1. The values given in parentheses are the percentage values of the profit in the corresponding column and π_1^*.

the one-cycle profits of all three models, assuming the optimal policies are applied. The numbers in parentheses are the percentage values of the particular setting relative to

[44]Approximately 63% of the profit is made in the first half of the planning horizon if prices and promotion are controlled dynamically. Also, about 64% of the total sales take place in this period.

the profit value of the dynamic case π_1^*, for example, the monopolist loses less than one percent of this total profit (the benchmark) if she is obliged (or chooses) to advertise at an (optimal) constant level. The last column represents the case when the price as well as the advertising rate is constant throughout the cycle. The optimal *fixed-price-fixed-advertising* pair is approximately given by $(18.29, 1, 493)$. The monopolist loses less than five percent of the total profit that she obtains if allowed to dynamically control both the price and the advertising rate. Similar observations can be made if *one* value of the original parameter setting in Example 2.4.1 changes as indicated in the first column of Table 2.2. Naturally, the values of π_1^* always exceed the other profit values. In this study, the drops are not *dramatic*; the profit values are relatively close. The largest deviation, see Table 2.2, occurs when we assume a (weekly) deterioration rate of five percent ($q(t) \equiv 0.05$). If the decision maker has the opportunity to dynamically control the price and advertising rate, she will indeed benefit from this option. If we talk about millions of dollars, five or ten percent more or less profit are substantial; and even in a small (family) business, 500 dollars per week add up to more than $25,000$ dollars per year.

Our results assume that the monopolist has complete foresight and full information about the parameters of the model. In practical applications, the cost structure is usually known, i.e., the values of c_0, ℓ, q, and r are reliable, but there is often uncertainty about the market parameters δ, ε, and μ. The functional expressions of the policies are robust with respect to parameter misspecifications. Assuming a wrong parameter (estimated) value and applying the suboptimal price-advertising strategy will naturally lead to a lower profit compared to the case when the true parameter value is known and the optimal policy is applied. However, in many cases the impact on the total profit is relatively small, although the suboptimal policy and the optimal policy might differ substantially. Panel (a) of Figure 2.8 shows the ratios of the total profit if the retailer applies a price-advertising strategy assuming $\varepsilon(t) \equiv 8$ and the total profit if the retailer applies the (optimal) price-advertising policy for the true price elasticity value according to the abscissa. We distinguish the four cases (i) *dynamic-pricing-dynamic-advertising*, (ii) *dynamic-pricing-static-advertising*, (iii) *static-pricing-dynamic-advertising*, and (iv) *static-pricing-static-advertising*. The results of cases (i) and (ii) hereby coincide almost exactly; the same is true about the results of cases (iii) and (iv). Around the value $\varepsilon = 8$ the deviations are relatively small. For example, in case (i), if the true price elasticity value is $\varepsilon(t) \equiv 6$, i.e., customers are less price sensitive than assumed, applying the (optimal) policy for $\varepsilon(t) = 8$ leads to a total profit that accounts for approximately



(a) (b)

Figure 2.8: (a) ratios of the total profit if the retailer applies a price-advertising strategy assuming $\varepsilon(t) \equiv 8$ and the total profit if the retailer applies the (optimal) price-advertising policy for the true price elasticity value according to the abscissa. (b) ratio of the total profit if the retailer applies a price-advertising strategy assuming $\mu_0 \equiv 20,000$ and the total profit if the retailer applies the (optimal) price-advertising policy for the true μ_0 value according to the abscissa. All other parameter values coincide with Example 2.4.1. (*i*) *dynamic-pricing-dynamic-advertising*, (*ii*) *dynamic-pricing-static-advertising*, (*iii*) *static-pricing-dynamic-advertising*, and (*iv*) *static-pricing-static-advertising*.

94 percent of the total profit associated with the truly optimal policy for $\varepsilon(t) = 6$.[45] If customers are more price sensitive, for example, $\varepsilon(t) = 10$ is the true value, the total profit associated with the suboptimal policy accounts for almost 95 percent of the optimal total profit.

Panel (b) of Figure 2.8 illustrates the results of the corresponding analysis with respect to changes in μ_0; the average number of *SCU*s bought. All the model settings, cases (*i*) to (*iv*), show essentially the same behavior: a (much) smaller true value of μ_0 reduces the profits relatively strong and a larger value of μ_0 reduces the profits only little if the (now) suboptimal price and advertising policy is applied. For instance, in Example 2.4.1, if the number of expected sales per week (at unit cost) doubles, i.e., $\mu_0 \equiv 40,000$, running the same advertising policy[46] as if μ_0 was equal to $20,000$ guarantees the monopolist still more than 96 percent of the profit she can optimally obtain when the true μ_0 value is known. Varying the values of other parameters or disregarding a time dependence of a parameter provides similar results: the optimal policies we derived in Sections 2.2 and 2.3 are robust with respect to misspecifications, i.e., if these policies are applied

[45] We neglect the effect that the retailer is committed with his order; instead, we assume that it is possible to reorder or return any amount at unit cost c_0.

[46] The optimal price policy is independent of μ and thus unaffected by changes of the μ_0 value.

using incorrect parameter (estimated) values, the resulting (suboptimal) total profits are relatively close to the optimal ones. This observation is not confined to our example. While one can always construct *extreme* scenarios where small differences between the estimated and the true value have great consequences with regard to the optimal profit, this robustness of the optimal policies is an essential feature of our model and the results. The following Remark summarizes our findings in the context of Example 2.4.1.

Remark 2.4.2 *For the parameter setting of Example 2.4.1 all four models* (*i*) *to* (*iv*), *see above, are robust with respect to parameter misspecifications. In particular, we make the following observations.*

- *A fixed-price strategy smooths the sales rate.*
- *Deviations from the optimal price entail larger drops in profits than using a suboptimal advertising rate.*
- *If parameter values are unknown, it is recommended to use conservative estimates. The parameter estimates can be updated over time.*
- *The larger the advertising efficiency parameter* Δ, *the larger are the benefits from dynamic pricing and dynamic advertising.*
- *The larger the demand elasticity parameter* ε, *the larger are the benefits from dynamic pricing and dynamic advertising.*

3 Optimal Dynamic Pricing and Advertising in New-Product Adoption Models

3.1 Introduction

The following framework provides a tool to analyze and understand controlled adoption models. The adoption of a new product by a market eventually mirrors the sales process of a (maybe very large) inventory. To this end, we occasionally switch between the terms adoption (model) and inventory (model). This variation will also remind the reader of the possible fields of application. A precise description of the model and assumptions is introduced in Section 3.2. Two main results therein, Theorems 3.2.1 and 3.2.2, have been published in Helmes et al. (2013). There, the authors apply the Dynamic Programming approach to derive and prove these theorems. In this thesis, we make use of the Maximum Principle to derive these results. In Section 3.3, we consider the particular class of *von Bertalanffy* adoption models to utilize and illustrate the implications of the main theorems.

To begin with, we present the general ideas of the model and a classification within the context of this work. Actually, we change perspectives: instead of looking from the perspective of a single firm we now take a broader market view. In Chapter 2, the models and the analysis were motivated from an individual company's point of view: faced with a specific cost structure and a particular market situation the retailer has to choose the pricing strategy, the advertising strategy, and the associated inventory capacity such that the total profit is maximized. One key assumption in Chapter 2 is that the level of the initial inventory is subject to the choice of the decision maker. This might be realistic in case of a (small) regional market where a retailer has full market power. But commonly, there exists a maximum quantity or total value of goods that can be sold, i.e., the market potential (of a single company) is limited to some positive value $\mathfrak{M}$, or more generally, one can think of a maximum quantity of sales that can be achieved by all companies in the market. Following the ideas of Chapter 2 the value $\mathfrak{M}$ is the (maximum) initial inventory or capacity - the maximum amount of goods that can be sold on the market. But instead of working with absolute values, we let $x(t)$ denote the *fractional* market share still to be captured at time t, $0 \leqslant t \leqslant T$. Naturally, $x(t)$ is a number between zero and one. While the normalization of x is only a matter of scaling, the main feature of the model analyzed in this chapter is that the initial value is now assumed to be fixed and no longer subject to the choice of the decision maker. Except for the cost of advertising $w(t)^a$, $a > \delta \geqslant 0$, no other cost terms are considered, i.e., the cost functional $c(t)$ introduced in Chapter 2, cf. (2.8), equals zero.[1] Instead, we now consider a state dependent demand, i.e., the demand rate at time t depends explicitly on the value of $x(t)$. We model this dependence by an additional factor $\psi(x)$ as part of the demand rate (2.4), i.e., $\lambda(t, p, w, x) = \mu(t)p^{-\varepsilon}w^{\delta}\psi(x)$, see below. With the help of the system function ψ one is able to model various effects. For example, if the function $\psi(x)$ decreases in x, the demand rate increases if more customers purchase the product.[2] In Chapter 2, we (implicitly) considered the special case where ψ is constant; we assumed this constant to be part of the μ function.

If ψ is non-constant, i.e., the demand at time t depends on the market share still to be captured, there is a *feedback* between the state $x(t)$ and the control that led to $x(t)$. While we still assume the demand rate to be zero if advertising spending is zero (in case $\delta > 0$), the actual market share (implicitly) *remembers* the expenditures on advertising up to now: if the product (or market) has been strongly promoted (and/or prices were set at a low level) one may expect the remaining market share to be *relatively* small. If none or only few advertising campaigns have been run, the main fraction of the market still waits to be captured. In the marketing literature this effect is often directly modeled by (so-called) *goodwill*. The money invested in promotion builds a sort of advertising capital (the *goodwill*). Nerlove and Arrow (1962) were one of the first to consider this framework; hence, the model is widely known as *Nerlove-Arrow* (advertising) model. Motivated by the empirical findings that the effect of advertising persists but diminishes over time Nerlove and Arrow assume the stock of goodwill to depreciate over time. Besides price and time, the rate of sales in their model depends on the level of goodwill and rather not on the current advertising spending. The idea is that goodwill attracts new customers or influences the preferences of customers. The depreciation of goodwill over time is motivated by the observation that consumers *drift* to other brands or (new) products. Considering only investments in goodwill and no price control, *Nerlove-Arrow* show that it is best to build up an optimal goodwill level

[1] Applications, where marginal costs or variable costs are zero are, for example, licensed software or electronic books, music, and movies distributed via the Internet.

[2] In the context of inventory control or shelf space management this corresponds to the situation when customers become more interested in purchasing a product if the number of available items is small, i.e., customers become impatient.

as fast as possible. In the infinite horizon problem the monopolist aims at this optimal level and only compensates the depreciation of goodwill; she will then advertise at a constant rate; Appendix 1 provides more details on the *Nerlove-Arrow* model.

One key property of the *Nerlove-Arrow* model is that the selling horizon and the amount that can be sold are unlimited as no market saturation occurs. A model that accounts for saturation effects is the classical model of Vidale and Wolfe (1957), cf. Appendix 2. Vidale and Wolfe were primarily interested in describing and explaining actual market behavior instead of controlling the demand rate in favor of some payoffs. They assume that the influence of advertising depends on two effects: the active response of the market (share) still to be captured and the forgetting of the share that has already been captured. The active response is modeled via the so-called *response constant*: the number of sales generated per invested dollar of advertising. Usually, we assume the response constant to be a factor of the more general function μ, cf. Chapter 2.[3] Our interpretation of such a factor diverges since w enters the dynamics and the objective as power expressions with exponent δ resp. a. However, the idea of a response constant in the model of Vidale and Wolfe carries over: one can think of a power form expression μ_1^δ to represent the response constant, cf. Section 2.1, p. 14. The effect of forgetting can be modeled via the system function ψ. While we assume that past expenditures on advertising have no lasting effect and only the current advertising rate influences the demand, the market memorizes past sales - which were influenced by past marketing activities - by means of the remaining market share. For example, if $\psi(x)$ is an increasing function, the demand rate declines with an increasing adoption of the product, ceteris paribus. In particular, if $\psi(x)$ tends to zero for small market share values, it needs a higher promotional effort (or a lower price) to draw the customers' attention to remaining items.

In the framework of marketing a new product or technology, the question of how and when such a new product or technology is adopted and diffused by the market has always been of particular interest. Influenced by contagion models in epidemiology and behavioral sciences, researchers set up models of adoption and diffusion that fit the empirical observations, see, for example, Mahajan et al. (1990) and Peres et al. (2010) for a review on new-product adoption literature.[4] One of the most important models is the one by Bass (1969).[5] Motivated and underpinned by empirical findings Bass finds that the process of new-product adoption for consumer durable goods - the classical examples

[3] Since the function μ depends on time this also allows for the response to be a time-dependent function.

[4] See also Huang et al. (2012) for a review on *Recent Developments in Dynamic Advertising Research*.

[5] Hopp (2004) selects the paper by Bass to be one of *Ten Most Influential Papers of Management Science's First Fifty Years*.

being refrigerators, (black & white) television, or room air conditioners in the United States - can be (quite well) described by a (simple) nonlinear differential equation, $t \geqslant 0$,

$$\dot{y}(t) = \Omega\,(1 - y(t)) + \Gamma y(t)\,(1 - y(t))\,, \quad y(0) = 0, \tag{3.1}$$

where $y(t) \in [0, 1]$ is the ratio of the cumulative sales relative to a fixed market potential $\mathfrak{M}$; $\Omega, \Gamma \geqslant 0$. The adoption and diffusion of a new product or technology depends on two forces: *innovation* (Ω) is proportional to $(1 - y(t))$ and *imitation* (Γ) is proportional to $y(t)\,(1 - y(t))$.[6] An innovator is not influenced by other (potential) customers' decisions or opinions, while an imitator's timing of adoption *is* influenced by the decision of others - the pressure of the social system or the so-called *word of mouth* effect. If more and more people adopt the product, the pressure increases on the remaining non-adopters. One immediate implication of the Bass model is that if no innovators appear ($\Omega \equiv 0$), the new product will not be adopted at all since $y(0) = 0$. While the values of both parameters determine the level of the (cumulative) sales rate, it is the ratio of both parameters that determines the behavior or shape of the adoption process. Note, if the external effects are shut down, i.e., $\Omega = 0$, and assuming $0 < y(0) < 1$, the Bass model is identical to the model proposed by Mansfield (1961). The empirical findings of Bass (1969) and others, see, for instance, Mahajan et al. (1995) or Marković and Jukić (2013), indicate that estimated values of Ω are relatively small, around 0.01 and often smaller, rarely exceeding 0.03. Estimates of the parameter Γ take values between 0.2 and 0.7. Mahajan et al. (1995) estimate, based on the results of Bass (1969), that 0.2 to 2.8 percent of all (potential) adopters are not influenced by the social system and adopt the product due to external effects.

Our framework is motivated by (optimal) inventory control models. Therefore, we will consider as state variable the *remaining* (fractional) market potential of a new product rather than the (fractional) cumulative sales, cf. below. Panel (a) of Figure 3.1 depicts the adoption rate and panel (b) of Figure 3.1 the remaining market potential $1 - y(t)$ at time t for different values of the parameter Ω; we set $\Gamma \equiv 1$. If the ratio of innovators to imitators is small, e.g., $\Omega/\Gamma = 0.001$, the adoption of the new product starts slowly. If the share of innovators is large ($\Omega = 0.5$), the market clears rapidly.

The *Generalized Bass Model* proposed by Bass et al. (1994) includes decision variables - price and adverting - to describe and control the adoption process. Sethi et al. (2008)

[6] More generally, these two effects are often described as *external* influence (innovation) and *internal* influence (imitation).

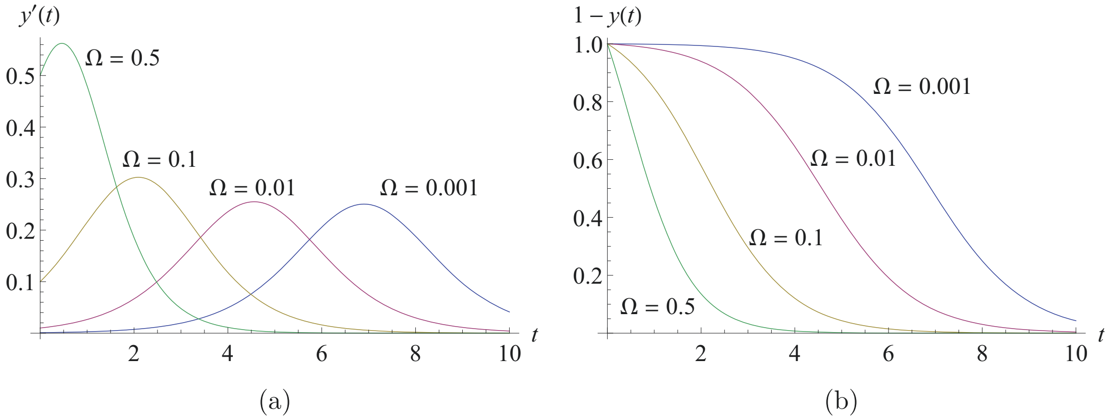


Figure 3.1: Bass model: growth rate of a new product (a) and remaining market potential (b) at time $t \in [0, 10]$ for different Ω values and $\Gamma = 1$.

consider a particular version of the Generalized Bass Model, where the Bass functional

$$\phi_{Bass}(y) = \Omega\,(1-y) + \Gamma y\,(1-y)\,, \quad y \in [0,1],$$

is approximated by the function $\sqrt{1-y}$ when $\Omega = \Gamma = 1$. Helmes et al. (2013) analyze the accuracy of this approximation and associated optimal controls. The state variable $y(t)$, $t \geqslant 0$, $y(0) = y_0 \in [0,1]$, denotes the fractional *market share* of a durable good, i.e., the cumulative sales relative to the absolute market potential of that product.[7] Control variables are price p and advertising rate w. Sethi et al. (2008) assume the deterministic demand rate is $\lambda_S(p,w,y) = \mu_S w f_S(p)\sqrt{1-y}$, where μ_S is a positive constant and $f_S(p)$ is a function of the price p. The authors consider the function f_S to be either of the linear form $f_S(p) = (1 - \varepsilon_S p)$, $0 \leqslant p \leqslant 1/\varepsilon_S$, $\varepsilon_S > 0$, or to be isoelastic of the form $f_S(p) = p^{-\varepsilon}, \varepsilon > 1$. The profit function to be maximized by choosing control functions $p(t)$ and $w(t)$ is

$$\int_0^\infty e^{-rt}\left[p(t)\lambda_S(t) - w(t)^2\right] dt, \tag{3.2}$$

subject to the state equation $\dot{y}(t) = \lambda_S(p(t), w(t), y(t))$, $y(0) = 0$. Except for the quadratic advertising costs, which effectively act as costs for *producing* y, no inventory costs, purchasing expenses, or other running costs are taken into account. In case of the isoelastic demand function, Sethi et al. (2008) show the optimal price p_S^* is constant in time, while the optimal advertising rate w_S^* is proportional to the approximating term

[7]In terms of inventory management $y(t)$ is the fraction of the total inventory sold at time t.

$\sqrt{1-y}$, see below, $\alpha_S = const$:

$$p_S^* = \varepsilon/(\varepsilon-1)\alpha_S, \quad \text{and} \quad w_S^*(y) = \frac{\mu_S}{2\varepsilon}\left(\frac{\varepsilon}{\varepsilon-1}\alpha_S\right)^{-(\varepsilon-1)}\sqrt{1-y}.$$

The value function is of the simple form $V_S(x) = \alpha_S\, y$.

The (uncontrolled) Bass model has been found to fit empirical adoption processes in many fields of application. Nevertheless, the basic structure of the model has been criticized as being too rigid. To overcome this criticism, Mahajan et al. (1990) characterize a diffusion model in terms of the *point of inflection* of the aggregate sales function - the point in time where the diffusion rate peaks - and in terms of *symmetry*. For the uncontrolled Bass model the point of inflection T_{Bass} can only occur before the product has captured half of its market potential.[8] Furthermore, the diffusion curve in the Bass model is symmetric around the peak time T_{Bass} in the interval $[0, 2T_{Bass}]$, cf. panel (a) of Figure 3.1. Mahajan et al. (1990), p. 10, claim that "In practice as well as in theory, the maximum rate of diffusion of an innovation should be able to occur at any time during the diffusion process. Additionally, diffusion patterns can be expected to be nonsymmetric as well as symmetric." A model that allows for such flexibility has been suggested by Easingwood et al. (1983). They assume a nonuniform influence (NUI) of the word of mouth (the external or imitation effect) on the evolution of the state y, i.e., $\Theta \geqslant 0$,

$$\phi_{NUI}(y) = \left(\Omega + \Gamma y^{\Theta}\right)(1-y).$$

However, in contrast to the uncontrolled Bass model no closed-form solution of the differential equation $y'(t) = \phi_{NUI}(y(t))$ exists in the NUI model. A diffusion model that can be solved explicitly has been proposed by Von Bertalanffy (1957) in the (biological) context of metabolism and growth. Von Bertalanffy (vB) assumes that the growth rate of the body weight y of an animal may be expressed as, $\theta \geqslant 0, \theta \neq 1, \Gamma \geqslant 0$, $0 < y < 1$,

$$\phi_{vB}(y) = \frac{\Gamma}{1-\theta} y^{\theta}\left(1 - y^{1-\theta}\right). \tag{3.3}$$

The parameters Γ and θ are constants related to *anabolism* and *catabolism*. In the context of new-product diffusion the quantity y is the fractional market share and the parameter Γ has the same interpretation as the *internal* influence (word of mouth) coefficient in the Bass model. Due to the parameter θ, however, the word of mouth effect changes over time in accordance with changes of the market fraction. As a result,

[8] The peak time is given by $T_{Bass} = -\frac{1}{\Gamma+\Omega}\log\left(\frac{\Omega}{\Gamma}\right)$, and the remaining market potential at that point equals $y\left(T_{Bass}\right) = \frac{1}{2} + \frac{\Omega}{2\Gamma} > \frac{1}{2}$.

the aforementioned flexibility in the characteristics of the diffusion curve is guaranteed. In contrast to the function ϕ_{Bass}, the expressions of ϕ_{vB} that depend on Γ and θ are not split into two distinct addends. Instead, these expressions are multiplied. Panel

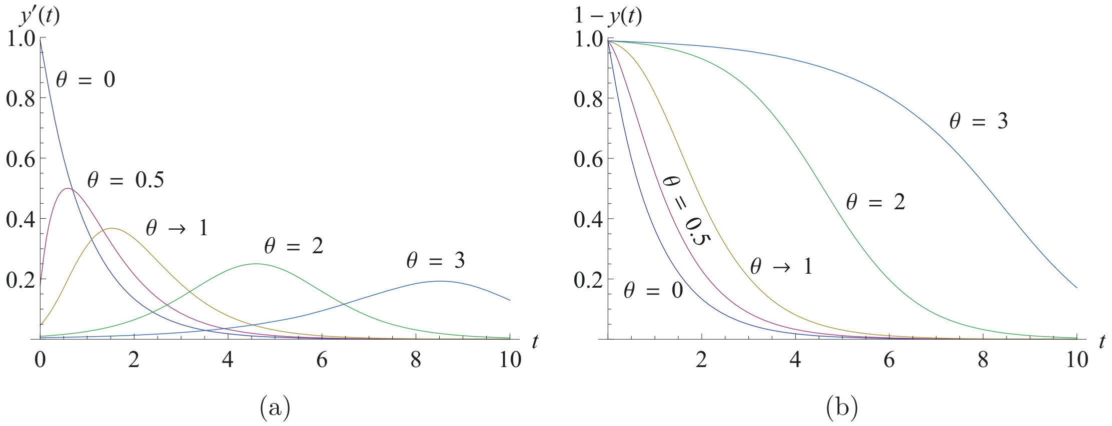


Figure 3.2: Von Bertalanffy model: growth (sales) rates of a new product (a) and remaining market potentials (b) at time $t \in [0, 10]$ for different θ values and $\Gamma = 1$, $y(0) = 0.01$.

(a) of Figure 3.2 illustrates the (uncontrolled) growth rate ϕ_{vB} for various θ values. Panel (b) of Figure 3.2 shows the associated evaluations of the remaining market shares defined by $1 - y(t)$, $y'(t) = \phi_{vB}(y(t)), y(0) = 0.01, t \geqslant 0$. Notice that there must be a positive number $y(0)$ of sales at hand to *start* the adoption process, since, if $y(0) = 0$, the process remains at zero.[9] If $\theta \equiv 0$, the function ϕ_{vB} reduces to $\Gamma(1-y)$. This expression strictly decreases in y, i.e., the more customers adopt the product the smaller the rate of adoption becomes. The adoption rate peaks at $T_{vB} = 0$ and monotonically decreases over time. If $\theta \equiv 2$, the model reduces to the model proposed by Mansfield (1961), $\phi_{Mansfield}(y) = \Gamma y(1 - y)$. In Mansfield's model, the growth rate is symmetric around the point of inflection T_{vB}, see below. Generally, panel (a) of Figure 3.2 suggests that the growth rate is skewed to the right whenever θ lies in the interval $[0, 2)$, i.e., the sales rate peaks when less than half of the (potential) market share has been acquired. If $\theta > 2$, the growth rates are skewed to the left: the maximum rate of sales is reached after more than half of the overall market volume has been covered. In general, the point of inflection - the point in time when the sales rate peaks - is given by, $y(0) \in (0, 1), \theta \geqslant 0, \theta \neq 1$,

[9] In the controlled model, when we consider the rate of adoption to depend on price and advertising rate as well, the promotion acts as an external influence on customers. Thus, the case $y(0) = 0$ will also be considered.

$\Gamma > 0$,

$$T_{vB} = \begin{cases} 0, & \text{if } y(0)^{1-\theta} > \theta, \\ -\frac{1}{\Gamma}\log\left(\frac{1-\theta}{1-y(0)^{1-\theta}}\right), & \text{else.} \end{cases}$$

If θ approaches 1, the model reduces to the Gompertz model characterized by the function $\phi_{Gompertz}(y) = \Gamma y \log(1/y)$, cf. Hendry (1972) and Dixon (1980). In the case of the Gompertz curve the point of inflection equals $1/\Gamma \log(-\log(y(0)))$.

Motivated by the inventory control problem in Chapter 2, from now on, we will rephrase the diffusion problem in terms of the variable $x := 1 - y$. Thus, x denotes the (fractional) market potential or (fractional) inventory that is still *available*. Using the variable x, the system function ψ in the von Bertalanffy model becomes, $\theta \geqslant 0, \theta \neq 1, \Gamma \geqslant 0$, $0 \leqslant x \leqslant 1$,

$$\psi(x) = \psi_{vB}(x) = \phi_{vB}(1-y) = \frac{\Gamma}{1-\theta}\Big[(1-x)^{\theta} - (1-x)\Big].$$

Taking up the ideas of Chapter 2, we will also consider time-dependent arrival intensities $\mu(t)$. Again, our goal is to determine optimal price and advertising policies over finite and infinite time horizons such that the profit, cf. (3.2), is maximized. The problem and the model are described in Section 3.2. In the same section, we present the solution of the general control problem. In Section 3.3, we analyze the controlled von Bertalanffy model. Unless otherwise stated the notation is the same as the notation used in Chapter 2. This includes the meaning and possible interpretation of parameters and variables.

3.2 The Model

We consider a monopolist who is selling goods during a time period T, $T > 0$; we allow T to be finite or infinite. The state variable $x(t)$ is the inventory at hand at time t, $0 \leqslant t \leqslant T$. Alternatively, x can be interpreted as the untapped market share of a durable good, cf. Section 3.1. From now on, we assume $x(t)$ to represent a fraction of a maximum amount $\mathfrak{M}$, i.e., $x(t) \in [0, 1]$. The actual market share values are easily obtained by multiplying the fraction by $\mathfrak{M}$. The initial (fractional) market size (inventory capacity) is given by $x(0) = x_0, 0 < x_0 \leqslant 1$; it is *not* subject to the decision of the monopolist. For instance, the value $x_0 = 1$ indicates that the full market potential $\mathfrak{M}$ can be exploited. Backordering is not allowed, and it is not possible to tap more than the total market, i.e., $x(T) \geqslant 0$.[10] The two controls are the price $p(t)$ and the advertising rate $w(t)$ at

[10] Note, in Helmes et al. (2013) the state variable $y(t)$ denotes the untapped market share. Sethi et al. (2008) let x denote the (fractional) cumulative sales. In order to be consistent with the preceding notation, cf. Chapter 2, we define x as the remaining fractional market share.

time t, $0 \leqslant t \leqslant T$. Both variables influence the deterministic demand. The demand rate also depends on the (current) value of x through the additional factor $\psi(x)$; we call ψ the system function. For $x > 0$, we define

$$\lambda^{(u)}(t,x) := \lambda\left(t, x(t), u(t)\right) = \lambda\left(t, x(t), p(t), w(t)\right) = \mu(t) p(t)^{-\varepsilon} w(t)^{\delta} \psi(x(t)), \tag{3.4}$$

where ε is the price elasticity and δ denotes the advertising elasticity; we set $\lambda^{(u)}(t,0) = 0$ for all $t \in [0,T]$. We assume $1 < \underline{\varepsilon} < \varepsilon < \infty$ and $0 \leqslant \delta < a < \overline{a}$, where a is the advertising cost parameter.[11] The function $\mu(t) > 0$, the arrival intensity, captures the *basic* demand, i.e., time-dependent influences such as seasonal effects, and all effects determined by the market which are not subject to the decision of the monopolist, cf. Chapter 2. The system function ψ is assumed to be positive and differentiable on the open interval $(0, x_0)$; the function $\psi^{1/(\varepsilon-1)}$ is assumed to be integrable on the open interval $(0, x_0)$. Note, if $\psi(x) \equiv 1$, the demand rate $\lambda^{(u)}(t,x)$ does not depend on the state and is equivalent to $\lambda^{(u)}(t)$, see (2.4). The monopolist has to choose a strategy u from the set of feasible policies U, see below, such that the ordinary differential equation (ODE), $x_0 > 0$,

$$\dot{x}(t) = -\lambda^{(u)}(t,x), \qquad x(0) = x_0,\ x(T) \geqslant 0, \tag{3.5}$$

is satisfied. For any s, $0 \leqslant s \leqslant T$, the set of feasible policies is given by

$$U(s,x) = \left\{ u \;\middle|\; \begin{array}{l} u(t) = (p(t), w(t))\,, \text{ and } u(t) \text{ is a vector-valued piecewise continuous} \\ \text{function on } [s,T], s \leqslant t \leqslant T, \text{ such that the solution of (3.5) with} \\ \text{initial condition } x(s) = x \text{ and terminal condition x(T)} \geqslant 0 \text{ is} \\ \text{uniquely determined and all integrals to be encountered in the} \\ \text{sequel exist; moreover, } p(t) > 0, w(t) \geqslant 0, t < T \end{array} \right\}.$$

By definition, we only consider controls such that the state process is nonnegative. The decision problem is to choose a control $u(t) = (p(t), w(t)) \in U(0, x_0), 0 \leqslant t \leqslant T$, that maximizes

$$J(u) := J(u; 0, T, x_0) := \int_0^T e^{-R(t)} \left[p(t)\lambda^{(u)}(t, x(t)) - w(t)^a \right] dt, \tag{3.6}$$

where $R(t) = \int_0^t r(s)ds$ is the cumulative discount rate, cf. Chapter 2. We will call any such policy an optimal open-loop policy. Compared with the objective function

[11] In contrast to the dynamic model in Chapter 2 we assume the elasticity parameters δ and ε as well as the cost parameter a to be constant over time.

(2.6) of the problem which we considered in Chapter 2, the objective $J(u)$ involves no cost terms except the costs of advertising (and discounting); the revenue parts of both objective functions, price multiplied by demand, are identical. We focus on the influence of the system function - the market diffusion - and we discard storage and running costs. Moreover, the initial value x_0 is - in contrast to the problem considered in Chapter 2 - fixed and it is not a decision variable. Thus, the (total) production costs $c_0 x_0$ are fixed and can be disregarded as well. Also, we do not consider an additive term $-q(t)x(t)$ in the state equation, i.e., the deterioration rate of the market share is zero, $q(t) \equiv 0$.

We solve the maximization problem by applying Pontryagin's maximum principle. Given an optimal open-loop policy we deduce an optimal *feedback* control $\hat{u} := \hat{u}(t, x)$, $0 \leqslant t \leqslant T, 0 < x \leqslant x_0$, see below. In Helmes et al. (2013), we first derive the optimal feedback control by solving the Hamilton-Jacobi-Bellman equation and then deduce the optimal control in open-loop form. In practical applications, both control forms, open-loop and feedback, are important. The open-loop characterization gives the decision maker a *plan* how to control the system over time. The feedback form enables the decision maker to react to deviations and disturbances of the system.

We define the Hamiltonian function, $u = (p, w)$, as

$$H(t, x, u, \kappa) = e^{-R(t)} \left(p\lambda(t, x, p, w) - w^a\right) - \kappa\lambda(t, x, p, w), \tag{3.7}$$

where $\kappa \in \mathbb{R}$ is called the adjoint variable.[12] Let $u^\star(t) = (p^\star(t), w^\star(t)) \in U$ be a control that satisfies (3.5) and maximizes (3.6). For the time being, we assume that such an optimal open-loop policy exists; let $x^\star(t)$ denote the trajectory associated with this optimal control, i.e., the solution of the ODE (3.5) using $u^\star$.[13] The necessary conditions for $u^\star$ and $x^\star$ to be optimal are, see, for example, Chapter 2 in Seierstad and Sydsæter

[12]In the general optimization literature, the Hamiltonian function is defined more generally as $H(t, x, u, \kappa) = \kappa_0 e^{-R(t)} \left(p\lambda(t, x, p, w) - w^a\right) - \kappa\lambda(t, x, p, w)$, where κ_0 is a constant. Seierstad and Sydsæter (1987) note on page 86 that in "... the economic literature dealing with optimal control theory it is quite common to see, without justification, the assumption ..." $\kappa_0 = 1$; they refer to $\kappa_0 = 0$ as *abnormal* case. Assuming $\kappa_0 = 0$ means that the (integrand of the) objective function has *no influence* on the solution of the problem.

[13]Throughout Chapter 3 we will denote the optimal open-loop policy by a '$\star$' superscript. This does not refer to the results of Chapter 2, where we used a '$*$' superscript.

(1987), $0 \leqslant t \leqslant T$,

$$H\left(t, x^{\star}(t), u^{\star}(t), \kappa(t)\right) \geqslant H\left(t, x^{\star}(t), u, \kappa(t)\right) \qquad \text{for all } u \in U, \tag{3.8}$$

$$\dot{\kappa}(t) = -\frac{\partial H}{\partial x}\left(t, x^{\star}(t), u^{\star}(t), \kappa(t)\right), \tag{3.9}$$

$$\kappa(T)\, x^{\star}(T) = 0, \tag{3.10}$$

$$\kappa(T) \geqslant 0. \tag{3.11}$$

Equation (3.9) holds except at the points of discontinuity of $u^{\star}(t)$. The Hamiltonian is differentiable with respect to p and w since the demand rate λ is differentiable in p and w; the derivatives of λ with respect to p and w are identical to those in Chapter 2, namely $\partial\lambda(\cdot)/\partial p = -\varepsilon\lambda(\cdot)/p$, and $\partial\lambda(\cdot)/\partial w = \delta\lambda(\cdot)/w$, cf. the proof of Theorem 2.2.1.

The adjoint variable $\kappa(t)$ is the per unit change in the objective function (3.6) for a *small* change in $x(t)$ at time t, i.e., $\kappa(t)$ is the *shadow price* of one additional (inventory or market share) unit of x at time t. In particular, $\kappa(0)$ is the marginal rate of change of the objective J with respect to a change in the initial market potential x_0.[14] An economic interpretation of the Hamiltonian can be obtained by multiplying (3.7) by dt and making use of the fact that the state equation (3.5) implies $-\lambda(\cdot)dt = \dot{x}dt = dx$; thus,

$$H(t, x, u, \kappa)dt = e^{-R(t)}\left(p\lambda(t, x, p, w) - w^{a}\right)dt + \kappa dx. \tag{3.12}$$

The first term on the right-hand side of (3.12) is the contribution to the objective $J(u)$ from time t to $t + dt$ if the untapped market share is x and the control u is applied in the (small) interval $[t, t + dt]$. The differential $dx = -\lambda(\cdot)dt$ represents the change in the untapped market share from time t to $t + dt$ if the control u is applied; it is a nonpositive value since $\dot{x} = -\lambda$. Hence, the second term on the right-hand side of (3.12), κdx, is the (negative) value associated with the change in the market share in the interval $[t, t + dt]$. It is the opportunity cost of the monopoly for selling today instead of tomorrow. Thus, $H(t, x, u, \kappa)dt$ is the net contribution to the objective $J(u)$ from time t to $t + dt$ if the untapped market (share) value is x and the control u is applied: the direct contribution from the market share gained plus the indirect contribution from lost sales in the future. All these values are given in terms of *time-zero dollars*, for instance, $\kappa(t)$ is the shadow price of an additional unit x measured at time 0. To obtain the current value one simply

[14] Here, it makes sense to think of absolute market potential instead of fractions, so a neat interpretation can be obtained by multiplying the value x by $\mathfrak{M}$.

has to multiply the expression of interest by $e^{R(t)}$.

Let $\lambda^\star$ denote the sales rate evaluated along the optimal control $u^\star$ and the associated optimal path $x^\star$, i.e., $\lambda^\star(t) = \lambda(t, x^\star(t), u^\star(t))$. Then, condition (3.8) implies the first order conditions

$$\frac{\partial H}{\partial p}(t, x^\star(t), u^\star(t), \kappa(t)) = e^{-R(t)}\left(\lambda^\star(t) - \varepsilon\lambda^\star(t)\right) + \kappa(t)\varepsilon\frac{\lambda^\star(t)}{p^\star(t)} \overset{!}{=} 0, \tag{3.13}$$

and, if $w^\star(t) > 0$,

$$\frac{\partial H}{\partial w}(t, x^\star(t), u^\star(t), \kappa(t)) = e^{-R(t)}\left(\delta p^\star(t)\frac{\lambda^\star(t)}{w^\star(t)} - aw^\star(t)^{a-1}\right) - \kappa(t)\delta\frac{\lambda^\star(t)}{w^\star(t)} \overset{!}{=} 0. \tag{3.14}$$

It follows directly from equation (3.13) that the optimal price must satisfy

$$p^\star(t) = \frac{\varepsilon}{\varepsilon - 1}e^{R(t)}\kappa(t). \tag{3.15}$$

In Chapter 2, the optimal price p^* is a markup on the cost function $c(t)$. Here, no costs except the advertising cost are explicitly considered and the optimal price $p^\star$ is a markup on the opportunity cost represented by the future value of the adjoint variable, the shadow price of one unit of x at time t. The size of the markup is still uniquely determined by the value of the price elasticity. Rewriting equation (3.14) we obtain

$$w^\star(t)^a = \Delta\lambda^\star(t)\left(p^\star(t) - e^{R(t)}\kappa(t)\right) \overset{(3.15)}{=} \frac{\Delta}{\varepsilon - 1}e^{R(t)}\kappa(t)\lambda^\star(t), \tag{3.16}$$

where $\Delta = \delta/a$ is the advertising efficiency parameter. We assume $0 \leqslant \delta < a$, see below; whenever the limit $\Delta \to 0$ is considered, we keep the value of $a > 0$ fixed and let $\delta \to 0$. If not otherwise specified, we always assume $\delta > 0$ and $\Delta > 0$. It is an immediate consequence of (3.15) and (3.16) that an optimal pair $(p^\star(t), w^\star(t))$ must satisfy the dynamic Dorfman-Steiner relation (2.14):

$$\frac{w^\star(t)^a}{p^\star(t)\lambda^\star(t)} = \frac{\Delta}{\varepsilon}. \tag{3.17}$$

At each point in time it is optimal to keep the ratio of advertising spending to revenue equal to the same constant. This relationship is not only true for every t pointwise. Also, for the optimal control, the quotient of cumulative advertising cost and cumulative revenue is constant. Hence, on any subinterval of $[0, T]$ the cumulative optimal promotional effort is a fraction of the optimal revenue, specifically, $\int_0^T w^\star(t)^a dt = \frac{\Delta}{\varepsilon}\int_0^T p^\star(t)\lambda^\star(t)dt$; recall, $0 < \Delta < 1 < \varepsilon$. Before we display explicit solution expressions of $p^\star$, $w^\star$, and other quantities of interest, we define the time-to-go potential $A^{(0)}$, the future potential A, and the diffusion potential B. These potentials will enable us to formulate the solution of our control problem in a convenient way. Note, the parameter restrictions are identical to those in Chapter 2.

Definition 3.2.1 *Let $1 < \underline{\varepsilon} < \varepsilon$, $0 \leqslant \delta < a < \overline{a}$, $\Delta = \delta/a$, and $\gamma = \frac{\varepsilon-\Delta}{1-\Delta}$. Let $T > 0$, and let $r(t) \geqslant 0$ and $\mu(t) > 0$ be piecewise continuous functions on $[0,T]$. Let*

$$\eta(t) := \frac{\varepsilon - \Delta}{\Delta}\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon}\frac{\Delta}{\varepsilon-1}\mu(t)\right]^{\frac{1}{1-\Delta}}.$$

(a) We call $A^{(0)}(t), t \in [0,T]$,

$$A^{(0)}(t) := A^{(0)}(t,T) := \int_t^T e^{-\gamma R(s)}\eta(s)ds, \tag{3.18}$$

the time-to-go potential, and $A(t), t \in [0,T]$,

$$A(t) := A(t,T) := e^{\gamma R(t)}A^{(0)}(t,T),$$

the future potential.

(b) Let the function ψ be differentiable and take only positive values on $(0, x_0)$ such that $\psi(x)^{\frac{1}{\varepsilon-1}}$ is integrable on any subinterval $(0,x), 0 < x \leqslant x_0$. We define the diffusion potential as

$$B(x) := \frac{\gamma}{\gamma-1}\int_0^x \psi(z)^{\frac{1}{\varepsilon-1}}dz.$$

(c) Let $\hat{V}(t,x)$ denote the value function when x is the untapped market share at time t, i.e.,

$$\hat{V}(t,x) := \sup_{u\in U(t,x)}\left\{e^{R(t)}\int_t^T e^{-R(s)}\left(p(s)\lambda(s,x(s),u(s)) - w(s)^a\right)ds\right\}. \tag{3.19}$$

We call $V_{x_0}(t)$ the continuation value at time t when the initial value of the untapped market share is x_0 and an optimal control $u^\star$ with associated path $x^\star$ is applied from time 0 to t, i.e.,

$$V_{x_0}(t) = \hat{V}(t, x^\star(t)). \tag{3.20}$$

Remark 3.2.1 *It follows from Definition 3.2.1 that $\eta(t) > 0$ for all $t \in [0, T]$; furthermore,*

$$\dot{A}^{(0)}(t) := \frac{\partial A^{(0)}}{\partial t}(t) = -e^{-\gamma R(t)}\eta(t) < 0,$$

and, by simple algebra,

$$\dot{A}(t) := \frac{\partial A}{\partial t}(t) = A(t)\left(\gamma r(t) + \frac{\dot{A}^{(0)}(t)}{A^{(0)}(t)}\right).$$

Moreover,

$$B'(x) := \frac{\partial B}{\partial x}(x) = \frac{\gamma}{\gamma - 1}\psi(x)^{\frac{1}{\varepsilon-1}}$$

is positive on $(0, x_0)$. Hence, the inverse function of B exists, and it is strictly increasing.

If $\mu(t) \equiv \mu$ and $r(t) \equiv r > 0$, then $\eta(t) \equiv \eta$, and

$$(i) \qquad A^{(0)}(t,T) = \eta e^{-\gamma r t}\frac{1 - e^{-\gamma r(T-t)}}{\gamma r} \to \begin{cases} \eta(T-t), & \text{if } r \to 0, \\ \frac{\eta}{\gamma r}e^{-\gamma r t}, & \text{if } T \to \infty; \end{cases} \tag{3.21}$$

$$(ii) \qquad A(t,T) = e^{\gamma r t}A^{(0)}(t,T) \to \begin{cases} \eta(T-t), & \text{if } r \to 0, \\ \frac{\eta}{\gamma r}, & \text{if } T \to \infty. \end{cases} \tag{3.22}$$

With these preparations at hand, we are able to solve the problem of maximizing $J(u)$ subject to (3.5). We will first present formulas of the optimal path and the optimal control, see (3.23) to (3.25); the proof of these formulas is given on the following pages.

Theorem 3.2.1 *Assume all conditions that underly Definition 3.2.1 hold. Then, the optimally controlled process $x^\star$ that satisfies (3.5) evolves according to the formula, $0 \leqslant t \leqslant T$,*

$$x^\star(t) = B^{-1}\left(B(x_0)\frac{A^{(0)}(t,T)}{A^{(0)}(0,T)}\right). \tag{3.23}$$

The price $p^\star$ and the advertising rate $w^\star$ that maximize (3.6) subject to (3.5) are given by, $0 < t < T$,

$$p^\star(t) = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{A^{(0)}(0,T)}{B(x_0)}\right)^{\frac{1}{\gamma}} e^{R(t)}\psi\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}, \tag{3.24}$$

and

$$w^\star(t) = \left[\frac{\Delta}{\varepsilon - \Delta}\left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}} \eta(t)e^{-(\gamma-1)R(t)}\right]^{\frac{1}{a}}. \tag{3.25}$$

Proof. First, we exploit the necessary optimality condition (3.9) to derive an expression of the adjoint variable $\kappa(t)$ in terms of the optimal path. Then, we make use of the state equation (3.5) to derive an explicit solution expression for the optimal path $x^\star$. The optimal policies $p^\star$ and $w^\star$ will be deduced from (3.13) and the Dorfman-Steiner relation. To verify that $u^\star = (p^\star, w^\star)$ is indeed optimal, we construct a function $V^\star(t,x)$ using the control $u^\star$ and we show that $V^\star(t,x)$ satisfies the Hamilton-Jacobi-Bellman equation, see below.

Throughout the proof we assume δ is positive, and thus $\Delta = \delta/a > 0$. The optimal solution when advertising has no effect, i.e., when $\delta = 0$, can easily be deduced by keeping $a > 0$ constant and by letting δ converge to zero. In the following, we frequently write "$(\cdot)$" to indicate a generic argument of a function in order not to overburden the notation and to concentrate on the important expressions. For the same reason, we occasionally omit the time index t.

To derive an expression for the adjoint function (co-state trajectory) we will start by looking at the derivative of the Hamiltonian with respect to x. By assumption, $\psi(x)$ is differentiable on the open interval $(0, x_0)$, see Definition 3.2.1. Hence, the demand rate λ is also differentiable (with respect to x) on $(0, x_0)$,

$$\frac{\partial \lambda}{\partial x}(t,x,p,w) = \mu(t) p^{-\varepsilon} w^{\delta} \psi'(x) = \lambda(t,x,p,w) \frac{\psi'(x)}{\psi(x)};$$

recall, $\psi(x) > 0$ on $(0, x_0)$. Thus, the derivative of the Hamiltonian with respect to x, $x \in (0, x_0)$, is given by

$$\frac{\partial H}{\partial x}(t,x,u,\kappa) = \left(e^{-R(t)} p - \kappa\right) \lambda(t,x,u) \frac{\psi'(x)}{\psi(x)}; \tag{3.26}$$

if, in addition, $\psi(x) > 0$ and $\psi'(x) \neq 0$ at the initial value $x = x_0$ and at the value $x = 0$, then (3.26) is well defined on the closed interval $[0, x_0]$. Formula (3.26) will be exploited below.

A very special (extreme) control is $w(t) \equiv 0$ for all $t \in (0, T)$ (together with an arbitrary feasible price function). For any such control no sales will take place, and the state remains in x_0. On the other hand, since $\dot{x} = -\lambda \leqslant 0$, once an x trajectory hits zero, it remains at 0. Let T_1 denote the last point in time when a controlled process satisfying the state equation (3.5) is in *state* x_0, i.e.,

$$T_1 := \max\{t \,|\, x(t) = x_0, x(t) \text{ satisfies } (3.5), 0 \leqslant t \leqslant T\}.$$

Similarly, let T_2 denote the first point in time a controlled process is at *zero*,

$$T_2 := \min\{t \mid x(t) = 0, x(t) \text{ satisfies } (3.5), 0 \leqslant t \leqslant T\}.$$

If no such value exists, we set $T_2 = +\infty$, and the (controlled) trajectory has a strictly positive value $x(T)$ at time T. The values T_1 and T_2 are well defined, since, by assumption, $x(t)$ is continuous and equals x_0 at time 0.

First, let us assume $T_2 \leqslant T$. Then, by construction, $x_0 > x(t) > 0$ for all $t \in (T_1, T_2)$, and the derivative (3.26) holds for all $t \in (T_1, T_2)$. Hence, condition (3.9), $\dot{\kappa}(t) = -\frac{\partial H}{\partial x}(t, x^\star(t), u^\star(t), \kappa(t))$, implies that the adjoint function and the optimal path $x^\star$ satisfy the equations

$$\dot{\kappa}(t) = -\left(e^{-R(t)}p^\star(t) - \kappa(t)\right)\lambda^\star(t)\frac{\psi'(x^\star(t))}{\psi(x^\star(t))} \overset{(3.15)}{=} -\kappa(t)\frac{\lambda^\star(t)}{\varepsilon - 1}\frac{\psi'(x^\star(t))}{\psi(x^\star(t))} \tag{3.27}$$

on the interval (T_1, T_2). Since the state equation (3.5) holds true, the optimal sales rate satisfies $\lambda^\star(t) = -\dot{x}^\star(t)$. Hence, the factor of $\kappa(t)$, see the second equation of (3.27), can be rewritten as

$$-\frac{\lambda^\star(t)}{\varepsilon - 1}\frac{\psi'(x^\star(t))}{\psi(x^\star(t))} = \frac{\dot{x}^\star(t)}{\varepsilon - 1}\frac{\psi'(x^\star(t))}{\psi(x^\star(t))} = \frac{\partial \log\left(\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}\right)}{\partial t}.$$

Substituting this last expression for the factor in (3.27), and dividing by κ, we obtain

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{\partial \log\left(\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}\right)}{\partial t}. \tag{3.28}$$

Since $\dot{\kappa}(t)/\kappa(t) = \partial(\log\kappa(t))\partial t$, integrating both sides of (3.28) from T_1 to an arbitrary t, $T_1 < t < T_2$, we get

$$\begin{aligned}\log(\kappa(t)) &= \log\left(\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}\right) + c_1 \\ &= \log\left(\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}\right) + \log\left(e^{c_1}\right) \\ &= \log\left(e^{c_1}\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}\right).\end{aligned}$$

Thus, applying the exponential function, on the interval (T_1, T_2) the adjoint function is given by

$$\kappa(t) = e^{c_1}\psi(x^\star(t))^{\frac{1}{\varepsilon-1}}. \tag{3.29}$$

The function κ is positive on (T_1, T_2), since ψ is positive on $(0, x_0)$ and $x^\star(t) \in (0, x_0)$ for $t \in (T_1, T_2)$. The value of the constant c_1 is determined by the terminal conditions (3.10) and (3.11), see below. According to (3.15) and (3.16), the optimal price satisfies $p^\star(t) = \frac{\varepsilon}{\varepsilon-1} e^{R(t)} \kappa(t)$ and the optimal advertising rate satisfies $w^\star(t) = \left[\frac{\Delta}{\varepsilon-1} e^{R(t)} \kappa(t) \lambda^\star(t)\right]^{1/a}$. Evaluating the sales rate along $(p^\star(t), w^\star(t))$ and the associated path $x^\star(t)$ on (T_1, T_2), we derive an expression for $\lambda^\star(t)$ which depends on the values of the optimal trajectory $x^\star$. Recall, $\Delta = \delta/a \in (0,1)$, $\gamma = \frac{\varepsilon-\Delta}{1-\Delta}$, and $\frac{\gamma-1}{\gamma} = \frac{\varepsilon-\Delta}{\varepsilon-1}$. Thus,

$$
\begin{aligned}
\lambda^\star(t) &= \mu(t) p^\star(t)^{-\varepsilon} w^\star(t)^{\delta} \psi\left(x^\star(t)\right) \\
&= \mu(t) \left(\frac{\varepsilon}{\varepsilon-1} e^{R(t)} \kappa(t)\right)^{-\varepsilon} \left(\frac{\Delta}{\varepsilon-1} e^{R(t)} \kappa(t) \lambda^\star(t)\right)^{\Delta} \psi\left(x^\star(t)\right) \\
&= \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon} \left(\frac{\Delta}{\varepsilon-1}\right)^{\Delta} \mu(t) \left[e^{R(t)} \kappa(t)\right]^{-(\varepsilon-\Delta)} \psi\left(x^\star(t)\right) \lambda^\star(t)^{\Delta} \\
&= \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon} \frac{\Delta}{\varepsilon-1} \left(\frac{\varepsilon-1}{\Delta}\right)^{1-\Delta} \mu(t) \left[e^{c_1+R(t)} \psi\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}\right]^{-(\varepsilon-\Delta)} \psi\left(x^\star(t)\right) \lambda^\star(t)^{\Delta} \\
&= \left(\frac{\Delta}{\varepsilon-\Delta} \eta(t)\right)^{1-\Delta} \left(\frac{\varepsilon-1}{\Delta}\right)^{1-\Delta} \left(e^{c_1+R(t)}\right)^{-(\varepsilon-\Delta)} \psi\left(x^\star(t)\right)^{1-\frac{\varepsilon-\Delta}{\varepsilon-1}} \lambda^\star(t)^{\Delta} \\
&= \left(\frac{\gamma-1}{\gamma} \eta(t)\right)^{1-\Delta} \left(e^{c_1+R(t)}\right)^{-(\varepsilon-\Delta)} \psi\left(x^\star(t)\right)^{-\frac{1-\Delta}{\varepsilon-1}} \lambda^\star(t)^{\Delta}.
\end{aligned}
$$

Note, $\lambda^\star(t) > 0$ and $0 < \Delta < 1$. Dividing by $\lambda^\star(t)^{\Delta}$ and taking the $1/(1-\Delta)$-th root the sales rate associated with $u^\star$ and $x^\star$ equals, $T_1 < t < T_2$,

$$
\lambda^\star(t) = \frac{\gamma-1}{\gamma} e^{-\gamma(c_1+R(t))} \eta(t) \psi\left(x^\star(t)\right)^{-\frac{1}{\varepsilon-1}}. \tag{3.30}
$$

Since, see Remark 3.2.1, $B'(x) = \dfrac{\gamma}{\gamma-1} \psi(x)^{\frac{1}{\varepsilon-1}}$, and $\lambda^\star(t) = -\dot{x}^\star(t)$, equation (3.30) is equivalent to

$$
\dot{x}^\star(t) = -e^{-c_1\gamma} \frac{e^{-\gamma R(t)} \eta(t)}{B'(x^\star(t))}. \tag{3.31}
$$

Since $\frac{\partial B(x^\star(t))}{\partial t} = B'(x^\star(t)) \dot{x}^\star(t)$ and $\dot{A}^{(0)}(t, T_2) = -e^{-\gamma R(t)} \eta(t)$, cf. Remark 3.2.1, multiplying (3.31) by $B'(x^\star(t))$ and integrating both sides of (3.31) with respect to $t \in [T_1, T_2]$, we obtain that $x^\star$ satisfies the identity

$$
B(x^\star(t)) = e^{-c_1\gamma} A^{(0)}(t, T_2) + c_2;
$$

the constant c_2 will be determined below. By assumption, the function ψ is positive, it is continuous on $(0, x_0)$, and the inverse of B exists, cf. Remark 3.2.1. Thus, the optimal path $x^\star$ is given by

$$x^\star(t) = B^{-1}\left(e^{-c_1\gamma}A^{(0)}(t, T_2) + c_2\right). \tag{3.32}$$

To determine the values of the constants c_1 and c_2, observe that $A^{(0)}(T_2, T_2) = 0$. Since, by definition, $x^\star(T_2) = 0$, the constant c_2 is determined by the equation $B^{-1}(c_2) = 0$. Thus, $c_2 = 0$. The initial condition $x^\star(T_1) = x_0$ implies

$$x_0 = B^{-1}\left(e^{-c_1\gamma}A^{(0)}(T_1, T_2) + c_2\right) = B^{-1}\left(e^{-c_1\gamma}A^{(0)}(T_1, T_2)\right).$$

Solving this equation for c_1 yields

$$c_1 = \frac{1}{\gamma}\log\left(\frac{A^{(0)}(T_1, T_2)}{B(x_0)}\right). \tag{3.33}$$

Using the information that c_2 is zero and that c_1 is given by (3.33), it follows from (3.32) that the optimal path on $[T_1, T_2]$ equals

$$x^\star(t) = B^{-1}\left(B(x_0)\frac{A^{(0)}(t, T_2)}{A^{(0)}(T_1, T_2)}\right). \tag{3.34}$$

From (3.29) and (3.33) we derive that, on the interval (T_1, T_2), the adjoint variable is

$$\kappa(t) = \left(\frac{A^{(0)}(T_1, T_2)}{B(x_0)}\right)^{\frac{1}{\gamma}}\psi\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}. \tag{3.35}$$

Formula (3.35) combined with (3.15) implies that an optimal price function is given by

$$p^\star(t) = \frac{\varepsilon}{\varepsilon-1}\left(\frac{A^{(0)}(T_1, T_2)}{B(x_0)}\right)^{\frac{1}{\gamma}} e^{R(t)}\psi\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}. \tag{3.36}$$

Substituting the right-hand side of (3.33) for c_1 in equation (3.30), we obtain the following formula for the optimal sales rate, $T_1 < t < T_2$,

$$\lambda^\star(t) = \frac{\gamma-1}{\gamma}\frac{B(x_0)}{A^{(0)}(T_1, T_2)}\frac{e^{-\gamma R(t)}\eta(t)}{\psi\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}} = -\frac{\dot{A}^{(0)}(t, T_2)}{A^{(0)}(T_1, T_2)}\frac{B(x_0)}{B'(x^\star(t))}. \tag{3.37}$$

By making use of the Dorfman-Steiner relation, we obtain the following formula for the optimal advertising (cost) rate, $T_1 < t < T_2$,

$$w^\star(t)^a = \frac{\Delta}{\varepsilon - \Delta}\left(\frac{B(x_0)}{A^{(0)}(T_1,T_2)}\right)^{\frac{\gamma-1}{\gamma}} \eta(t)e^{-(\gamma-1)R(t)}. \tag{3.38}$$

The formula for the optimal control and the formula for the associated sales rate are defined on the open interval (T_1, T_2), $0 \leqslant T_1 \leqslant T_2 \leqslant T$. We will now show that for the optimal control $T_1 = 0$ and $T_2 = T$. To do so, we compute the value of the objective function $J(u^\star; T_1, T_2, x_0)$; in the second line of the following transformations we exploit the Dorfman-Steiner relation and, afterwards, use (3.38) and Definition 3.2.1:

$$\begin{aligned}
J(u^\star; T_1, T_2, x_0) &= \int\limits_{T_1}^{T_2} e^{-R(t)}\left[p^\star(t)\lambda^\star(t) - w^\star(t)^a\right] dt \\
&= \int\limits_{T_1}^{T_2} e^{-R(t)}\left[\frac{\varepsilon}{\Delta} w^\star(t)^a - w^\star(t)^a\right] dt \\
&= \int\limits_{T_1}^{T_2} e^{-R(t)}\frac{\varepsilon-\Delta}{\Delta}\frac{\Delta}{\varepsilon-\Delta}\left(\frac{B(x_0)}{A^{(0)}(T_1,T_2)}\right)^{\frac{\gamma-1}{\gamma}} \eta(t)e^{-(\gamma-1)R(t)} dt \\
&= \left(\frac{B(x_0)}{A^{(0)}(T_1,T_2)}\right)^{\frac{\gamma-1}{\gamma}} \int\limits_{T_1}^{T_2} e^{-\gamma R(t)}\eta(t) dt \\
&= \left(\frac{B(x_0)}{A^{(0)}(T_1,T_2)}\right)^{\frac{\gamma-1}{\gamma}} A^{(0)}(T_1,T_2) \\
&= A^{(0)}(T_1,T_2)^{\frac{1}{\gamma}} B(x_0)^{\frac{\gamma-1}{\gamma}}.
\end{aligned}$$

Since $\eta(t)$ is positive on the interval $[0,T]$, the function $A^{(0)}(T_1,T_2) = \int_{T_1}^{T_2} e^{-\gamma R(t)}\eta(t)dt$ is decreasing in T_1, and it is increasing in T_2, $0 \leqslant T_1 \leqslant T_2 \leqslant T$. Hence, $J(u^\star; T_1, T_2, x_0)$ attains its largest value at $T_1 = 0$ and $T_2 = T$.

So far we only analyzed the scenario $x^\star(T) = 0$, and we showed that among all controls such that $x^\star(T_2) = 0$ the one where $T_2 = T$ maximizes the objective. However, we also have to analyze the scenario where the terminal state is strictly positive. If $x^\star(T) > 0$, the preceding analysis holds true on the semi-open interval $(T_1, T]$. Note, $\psi(x)$ and $\psi'(x)$ are well defined at $x = x^\star(T) > 0$ and, according to (3.29), we can evaluate $\kappa(t)$ at $t = T$. Condition (3.10) requires that

$$x^\star(T)\kappa(T) = x^\star(T)e^{c_1}\psi\left(x^\star(T)\right)^{\frac{1}{\varepsilon-1}} = 0. \tag{3.39}$$

By assumption, $x^\star(T)$, $\psi(x^\star(T))$, and $\kappa(T)$ are positive. These facts contradict (3.39). Hence, a control such that $x^\star(T) > 0$ can not be optimal, and the scenario $x^\star(T) = 0$ is the only feasible one. Formulas (3.23) to (3.25) of Theorem 3.2.1 follow from equations (3.34), (3.36), and (3.38) by setting $T_1 = 0$ and $T_2 = T$.

Up to now, we only showed that $p^\star$, $w^\star$, and the associated path $x^\star$, cf. (3.23) to (3.25), satisfy the *necessary* optimality conditions. To prove that the open-loop control $u^\star = u^\star_{x_0}$ is optimal when maximizing (3.6) with the initial data $(0, x_0)$, we will show the following: for any pair $(t, x), 0 < t < T, x \in \mathbb{R}$, there exists an initial value $y_0 = y_0(t, x)$ such that $V^\star(t, x) := J\left(u^\star_{y_0}; t, T, y_0(t, x)\right)$ satisfies a (particular) Hamilton-Jacobi-Bellman equation, see below. Moreover, we will verify that the values $u^\star(t) = u^\star_{x_0}(t)$ coincide with the argmax values of the right-hand side of that HJB-equation.

To this end, let us assume that the system starting in x_0 is in state $x^\star(t)$ at time $t \in [0, T)$ and that the control $u^\star$ has been applied. The *continuation* value $V^\star_{x_0}(t)$, cf. (3.20), is given by

$$
\begin{aligned}
V^\star_{x_0}(t) &:= \int_t^T e^{-R(s)} \left(p^\star(s)\lambda^\star(s) - w^\star(s)^a\right) ds \\
&= \int_t^T e^{-R(s)} \left(\frac{\varepsilon}{\Delta} w^\star(s)^a - w^\star(s)^a\right) ds \\
&= \int_t^T e^{-R(s)} \frac{\varepsilon - \Delta}{\Delta} \frac{\Delta}{\varepsilon - \Delta} \left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}} \eta(s) e^{-(\gamma-1)R(s)} ds \\
&= \left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}} \int_t^T e^{-\gamma R(s)} \eta(s) ds \\
&= \left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}} A^{(0)}(t,T); \qquad (3.40)
\end{aligned}
$$

in the second line we make use of the Dorfman-Steiner relation (3.17). Note, $V^\star_{x_0}(t)$ is given in terms of time zero dollars, and $x^\star(t)$, assumed to be optimal, is given by formula (3.23). Hence, in (3.40), we can replace $B(x_0)$ by $B(x^\star(t))\frac{A^{(0)}(0,T)}{A^{(0)}(t,T)}$. Thus,

$$
V^\star_{x_0}(t) = \left(\frac{B(x^\star(t))}{A^{(0)}(t,T)}\right)^{\frac{\gamma-1}{\gamma}} A^{(0)}(t,T) = A^{(0)}(t,T)^{\frac{1}{\gamma}} B(x^\star(t))^{\frac{\gamma-1}{\gamma}}. \qquad (3.41)
$$

By a proper choice of the initial value x_0, we find our *qualified guess* of a function $V^\star(t,x)$ for any pair (t,x), $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant x_0$. To be precise, if (t,x) is given, then choose x_0 such that $x^\star(t;x_0) = x$, where $x^\star(t;x_0)$ is given by (3.23). Observe, see (3.23), $B\left(x^\star(t)\right)\frac{A^{(0)}(0,T)}{A^{(0)}(t,T)} = B(x_0) \iff x_0 = B^{-1}\left(B\left(x^\star(t)\right)\frac{A^{(0)}(0,T)}{A^{(0)}(t,T)}\right)$. Thus, $V^\star(t,x) = A^{(0)}(t,T)^{\frac{1}{\gamma}}B(x)^{\frac{\gamma-1}{\gamma}}$; notice that $V^\star(t,0) = V^\star(T,x) = 0$ for all $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant x_0$. Moreover, $V^\star$ is differentiable in t and x, where

$$\dot{V}^\star(t,x) = \frac{\partial V^\star}{\partial t}(t,x) = \frac{1}{\gamma}\dot{A}^{(0)}(t)\left(\frac{B(x)}{A^{(0)}(t)}\right)^{\frac{\gamma-1}{\gamma}}$$

and

$$V^{\star\prime}(t,x) = \frac{\partial V^\star}{\partial x}(t,x) = \left(\frac{A^{(0)}(t)}{B(x)}\right)^{\frac{1}{\gamma}}\psi(x)^{\frac{1}{\varepsilon-1}}. \tag{3.42}$$

Next, we will show that for all (t,x), $0 < t \leqslant T, 0 < x \leqslant x_0$, $V^\star(t,x)$ satisfies the Hamilton-Jacobi-Bellman equation

$$0 = \dot{V}^\star(t,x) + \max_{u\in U(t,x)}\left\{H\left(t,x,u,V^{\star\prime}(t,x)\right)\right\}. \tag{3.43}$$

For any t and x, $0 < t \leqslant T, 0 < x \leqslant x_0$, let $u^0 = \left(p^0, w^0\right) = \left(p^0(t,x), w^0(t,x)\right)$ denote a control maximizing $H\left(t,x,u,V^{\star\prime}(t,x)\right)$ and let $\lambda^0 := \lambda^0(t,x)$ be the associated sales rate. The expressions of p^0 and w^0 are identical to the right-hand side of $p^\star$ and $w^\star$ with $V^{\star\prime}(t,x)$ instead of $\kappa(t)$, i.e.,

$$p^0(t,x) = \frac{\varepsilon}{\varepsilon-1}e^{R(t)}V^{\star\prime}(t,x), \tag{3.44}$$

and

$$w^0(t,x)^a = \frac{\Delta}{\varepsilon-1}e^{R(t)}V^{\star\prime}(t,x)\lambda^0(t,x).$$

Elementary but lengthy calculations show that

$$w^0(t,x)^a = \frac{\Delta}{\varepsilon-\Delta}\eta(t)\left[e^{R(t)}V^{\star\prime}(t,x)\right]^{-(\gamma-1)}\psi(x)^{\frac{1}{1-\Delta}}. \tag{3.45}$$

Moreover, u^0 satisfies the dynamic Dorfman-Steiner relation, i.e.,

$$p^0(t,x)\lambda^0(t,x) = \frac{\varepsilon}{\Delta}w^0(t,x)^a \quad \Leftrightarrow \quad \lambda^0(t,x) = \frac{\varepsilon}{\Delta}\frac{w^0(t,x)^a}{p^0(t,x)}. \tag{3.46}$$

Using (3.44) and (3.46) the maximized Hamiltonian function $H^0(t,x)$ can be written as

$$\begin{aligned}
H^0(t,x) &= H\left(t,x,u^0(t,x),V^{\star\prime}(t,x)\right) \\
&= e^{-R(t)}\left[p^0(t,x)\lambda^0(t,x) - w^0(t,x)^a\right] - V^{\star\prime}(t,x)\lambda^0(t,x) \\
&= e^{-R(t)}\left[\frac{\varepsilon}{\Delta}w^0(t,x)^a - w^0(t,x)^a\right] - V^{\star\prime}(t,x)\frac{\varepsilon}{\Delta}\frac{w^0(t,x)^a}{p^0(t,x)} \\
&= \frac{\varepsilon-\Delta}{\Delta}e^{-R(t)}w^0(t,x)^a - V^{\star\prime}(t,x)\frac{\varepsilon}{\Delta}\frac{w^0(t,x)^a}{\frac{\varepsilon}{\varepsilon-1}e^{R(t)}V^{\star\prime}(t,x)} \\
&= \left(\frac{\varepsilon-\Delta}{\Delta} - \frac{\varepsilon-1}{\Delta}\right)e^{-R(t)}w^0(t,x)^a \\
&= \frac{1-\Delta}{\Delta}e^{-R(t)}w^0(t,x)^a.
\end{aligned} \tag{3.47}$$

Next, in (3.47) we replace $w^0(t,x)^a$ by the right-hand side of (3.45). Finally, if we use (3.42) and the fact that $\dot{A}^{(0)}(t) = -e^{-\gamma R(t)}\eta(t)$, cf. Remark 3.2.1, we obtain

$$\begin{aligned}
H^0(t,x) &= \frac{1-\Delta}{\Delta}e^{-R(t)}\frac{\Delta}{\varepsilon-\Delta}\eta(t)\left[e^{R(t)}V^{\star\prime}(t,x)\right]^{-(\gamma-1)}\psi(x)^{\frac{1}{1-\Delta}} \\
&= \frac{1-\Delta}{\varepsilon-\Delta}e^{-\gamma R(t)}\eta(t)\left[\left(\frac{A^{(0)}(t)}{B(x)}\right)^{\frac{1}{\gamma}}\psi(x)^{\frac{1}{\varepsilon-1}}\right]^{-(\gamma-1)}\psi(x)^{\frac{1}{1-\Delta}} \\
&= \frac{1}{\gamma}e^{-\gamma R(t)}\eta(t)\left(\frac{B(x)}{A^{(0)}(t)}\right)^{\frac{\gamma-1}{\gamma}}\psi(x)^{\frac{1}{1-\Delta}-\frac{\gamma-1}{\varepsilon-1}} \\
&= -\frac{1}{\gamma}\dot{A}^{(0)}(t)\left(\frac{B(x)}{A^{(0)}(t)}\right)^{\frac{\gamma-1}{\gamma}} \\
&= -\dot{V}^\star(t,x).
\end{aligned}$$

Thus, $V^\star(t,x)$ satisfies the HJB equation (3.43) and hence $u^\star$, cf. (3.24) and (3.25), is an optimal control. ♦

In the proof of Theorem 3.2.1 we have derived several useful results that we summarize in the next paragraphs. The sales rate associated with the optimal control - the optimally controlled adoption process - is given by (3.37), where $T_1 = 0$ and $T_2 = T$. Formula (3.40) is the continuation value at time t, discounted to time $t = 0$, if the untapped initial market share is x_0 and the optimal control $u^\star$ is applied. Thus, the maximal value of the objective function (3.6) is given by $J(u^\star;0,T,x_0) = V^\star(0,x_0)$. The optimal continuation value $V_{x_0}(t)$ in terms of time t dollars was defined in Definition 3.2.1. Accordingly, formula (3.50) is easily obtained by multiplying the last line of (3.40) by the growth factor $e^{R(t)}$.

Corollary 3.2.1 *The sales rate $\lambda^\star$ associated with the optimal control $(p^\star(t), w^\star(t))$, cf. Theorem 3.2.1, is given by*

$$\lambda^\star(t) = \lambda\left(t, x^\star(t), u^\star(t)\right) = -\frac{\dot{A}^{(0)}(t,T)}{A^{(0)}(0,T)} \frac{B(x_0)}{B'(x^\star(t))}. \tag{3.48}$$

The optimal value $J^\star(T, x_0) := J(u^\star; 0, T, x_0)$ is given by

$$J^\star(T, x_0) = \left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}}, \tag{3.49}$$

and the optimal continuation value by

$$V_{x_0}(t) = e^{R(t)} A^{(0)}(t,T) \left(\frac{B(x_0)}{A^{(0)}(0,T)}\right)^{\frac{\gamma-1}{\gamma}}. \tag{3.50}$$

An important implication of Theorem 3.2.1 is that the whole market (share) will be tapped, i.e., $x^\star(T) = 0$, if T is finite. Moreover, as shown in the proof of Theorem 3.2.1, it is optimal to control the sales process such that the optimal path will hit zero for the first time at time T.[15] This property is a consequence of the special (isoelastic) demand rate and the fact that prices can be set arbitrarily low. It is possible to gain large shares of the market in a short time interval by setting appropriate prices. In particular, if $\psi(x)$ tends to zero should $x \to x_0$, then the optimal price converges to zero and the optimal adoption rate should tend to $+\infty$; the same holds true if, for $x \to 0$, the value of $\psi(x)$ tends to zero. Both characteristics can be observed in many applications for particular adoption processes, for example, $\psi_{Mansfield}(x) = \Gamma x(1-x)$ or $\psi_{vB} = \frac{\Gamma}{1-\theta}\left[(1-x)^\theta - (1-x)\right]$, cf. Section 3.1 and Section 3.3. Note, the optimal advertising rate $w^\star$ does only depend on the diffusion potential $B(x_0)$ but does not depend on the current value of $x^\star(t)$.

In the verification step of the proof of Theorem 3.2.1 we solve the HJB equation (3.43) for our - now verified - guess. We obtain $\hat{V}(t,x)$, the value function in terms of time t dollars, by multiplying $V^\star(t,x)$ by $e^{R(t)}$. The control $\hat{u}$ is the solution to the maximization problem which is part of (3.43). Therefore, u^0 is the optimal control in feedback form.

[15] Note, the term $A^{(0)}(t,T)$ in the expression of the optimal path, cf. (3.23), will only be zero at $t = T < \infty$, see Definition 3.2.1 and parameter restrictions therein.

Theorem 3.2.2 *Assume all conditions that underly Definition 3.2.1 hold. Then, the value function* $\hat{V}$ *is given by,* $0 \leqslant t < T$, $0 < x \leqslant x_0$,

$$\hat{V}(t,x) = A(t)^{\frac{1}{\gamma}} B(x)^{\frac{\gamma-1}{\gamma}}. \tag{3.51}$$

The optimal price $\hat{p}$ *and the optimal advertising rate* $\hat{w}$ *that satisfy* (3.19) *are given by*

$$\hat{p}(t,x) = \frac{\varepsilon}{\varepsilon-1}\left(\frac{A(t)}{B(x)}\right)^{\frac{1}{\gamma}} \psi(x)^{\frac{1}{\varepsilon-1}} \tag{3.52}$$

and

$$\hat{w}(t,x) = \left[\frac{\Delta}{\varepsilon-\Delta}\eta(t)\left(\frac{B(x)}{A(t)}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{a}}. \tag{3.53}$$

The sales rate $\hat{\lambda}$ *associated with the optimal feedback policy is given by*

$$\hat{\lambda}(t,x) = \frac{\eta(t)}{A(t)}\frac{B(x)}{\psi(x)} = -\frac{\dot{A}^{(0)}(t)}{A^{(0)}(t)}\frac{B(x)}{B'(x)}. \tag{3.54}$$

Proof. By the definition of $A(t,T)$, cf. Definition 3.2.1,

$$A(t,T)^{\frac{1}{\gamma}} = \left(e^{\gamma R(t)}A^{(0)}(t,T)\right)^{\frac{1}{\gamma}} = e^{R(t)}A^{(0)}(t,T)^{\frac{1}{\gamma}}.$$

Thus, formula (3.51) follows from evaluating $\hat{V}(t,x) = e^{R(t)}V^{\star}(t,x)$. The optimal price in feedback form is given by (3.36). Replacing $V^{\star\prime}(t,x)$ by the right-hand side of (3.42) we obtain (3.52). Similarly, the optimal advertising rate $\hat{w}$ is the result from substituting $V^{\star\prime}(t,x)$ in (3.38). The adoption rate associated with $\hat{u} = (\hat{p},\hat{w})$ follows by making use of the Dorfman-Steiner relation (3.46), $\hat{\lambda}(t,x) = \frac{\varepsilon}{\Delta}\frac{\hat{w}(t,x)^a}{\hat{p}(t,x)}$ and the fact that

$$\frac{\eta(t)}{A(t)} = \frac{e^{-\gamma R(t)}\eta(t)}{e^{-\gamma R(t)}A(t)} = -\frac{\dot{A}^{(0)}(t)}{A^{(0)}(t)},$$

cf. Remark 3.2.1. ♦

Remark 3.2.2 *In the proof of Theorem 3.2.1, the value of the policy* $u^0(t,x)$ *is given in terms of time* t *dollars, whereas the guess of the value function* $V^{\star}(t,x)$ *is given in terms of time zero dollars; note, the discount factor in the Hamiltonian at time* t *is* $e^{-R(t)}$. *The quantities* $\hat{V}(t,x)$, $\hat{p}(t,x)$, $\hat{w}(t,x)$, *and* $\hat{\lambda}(t,x)$ *in Theorem 3.2.2 are given in terms of the future potential* $A(t)$. *The quantities associated with the optimal open-loop control in Theorem 3.2.1, however, depend on the time-to-go potential* $A^{(0)}(t)$. *Since*

$A(t) = e^{\gamma R(t)} A^{(0)}(t)$ *one can easily switch between both representations.*

The solution formulas in both Theorems 3.2.1 and 3.2.2 are of separable form: the expressions $A^{(0)}$ and A only depend on the time (to go) while B and ψ only depend on the value of the untapped market share. The interest rate and the function μ are *responsible* for the dynamics of the advertising rate, see, for instance, equation (3.25): if $r(t) \equiv r$, and if $\mu(t) \equiv \mu$ so that $\eta(t) \equiv \eta$, it is optimal to advertise at a constant rate. The evolution of the optimal prices over time explicitly depends on the function ψ. The formula of the optimal feedback policy implies that the optimal price is proportional to $\hat{V}'$. Taking the derivative one can easily verify

$$\hat{p}(t, x) = \frac{\varepsilon}{\varepsilon - 1} \hat{V}'(t, x). \tag{3.55}$$

The optimal advertising rate $\hat{w}$ is proportional to $\dot{\hat{V}}$ and satisfies the equation

$$\hat{w}(t, x) = \left[\frac{\Delta}{\varepsilon - \Delta} \eta(t) \frac{\hat{V}(t, x)}{A(t)} \right]^{\frac{1}{a}} = \left(\frac{\Delta}{1 - \Delta} \frac{\eta(t)}{\dot{A}(t)} \dot{\hat{V}}(t, x) \right)^{\frac{1}{a}}. \tag{3.56}$$

The optimal control in open-loop form and the optimal control in feedback form have similar properties. The optimal price is a markup on the marginal value of an additional unit of inventory at time t; the advertising effort *follows* the arrival intensity μ. According to equation (2.18) in Chapter 2,

$$\nu^*(t) = e^{-R(t)} \frac{1 - \Delta(t)}{\Delta(t)} w^*(t)^{a(t)},$$

the optimal profit margin is proportional to the advertising spending, where the size of the proportionality factor is determined by the value of the advertising efficiency Δ. It is interesting to note that the value of the Hamiltonian (the net contribution in terms of present value to the objective at time t) is of the same form as ν^*, cf. (3.47). Despite the differences between the problems considered in Chapter 2 and Chapter 3 - a state dependent demand rate, deteriorating effects, a free or given initial inventory value, and so on - the present values of the *profit rates* ν^* and H^0 have very similar characteristics: both are proportional to the optimal advertising spending multiplied by the factor $\frac{1-\Delta}{\Delta}$. Similarly, we can rewrite the continuation value and the value function at time t in terms of the advertising effort. Elementary calculations show that

$$V_{x_0}(t) = \frac{\varepsilon - \Delta}{\Delta} \frac{A(t)}{\eta(t)} w^\star(t)^a \tag{3.57}$$

and

$$\hat{V}(t,x) = \frac{\varepsilon - \Delta}{\Delta} \frac{A(t)}{\eta(t)} \hat{w}(t,x)^a. \tag{3.58}$$

Written this way, we see that the total value at time t depends on the *current* advertising effort $w^\star(t)^a$ and $\hat{w}(t,x)^a$, and three factors: the first factor $\frac{\varepsilon-\Delta}{\Delta}$ can be interpreted as the weight of the influence of price-advertising-effects. This factor increases in ε and decreases in Δ. This dependence is somewhat counterintuitive; one would expect the profit to *decrease* in the price elasticity ε and to *increase* in the advertising efficiency Δ. However, since both parameters show up in the factors η, A, and the advertising costs, it is not possible to prove general statements about how the values (3.57) and (3.58) depend on ε and Δ.[16] The second factor of the product on the right-hand side of (3.58) is the ratio of the future potential and the value of the parameter function η. In the time-homogeneous case ($r \equiv 0$ and $\mu(t) \equiv \mu$), this ratio equals $A(t)/\eta = T - t$, i.e., the second factor equals the time-to-go. Since in this particular setting, cf. (3.24) and (3.52), the optimal advertising rate is constant, the (net) value of the problem is given by the time-to-go times current (constant) advertising spending, multiplied by the factor $\frac{\varepsilon-\Delta}{\Delta}$.

In Chapter 4, we will solve the problem how to choose the cycle length and the initial inventory assuming the control given in Theorem 3.2.1 is used. With regard to the analysis that follows in Chapter 4, we like to point out that the value function $\hat{V}$ is a homogeneous *Cobb-Douglas* function in terms of the potentials A and B. Moreover, there are special cases when the value function at time zero takes the form $\hat{V}(0,x) = const \cdot T^{\frac{1}{\gamma}} x^{\frac{\gamma-1}{\gamma}}$, such that T and x can be directly interpreted as the sales or production time and the sales or production volume, c.f. Section 4.3.

All formulas and expressions hold true for positive values T. Moreover, we will deduce the optimal policies and associated values in the time-homogeneous setting if the time horizon is infinite, i.e., we assume $r(t) \equiv r, \eta(t) \equiv \eta$, and $T \to \infty$. For the open-loop representation as well as for the feedback form of the optimal policies we will indicate the most relevant characteristics of the infinite horizon model by a '$_\infty$' subscript. Since $\lim_{t\to\infty} A(t) = \frac{\eta}{\gamma r}$, cf. Remark 3.2.1, the feedback policies will only depend on the current state, and not on time.

Corollary 3.2.2 *Take Definition 3.2.1, and let $r(t) \equiv r > 0$, $\eta(t) \equiv \eta$. Consider the infinite horizon problem, i.e., let $T \to \infty$ in Definition 3.2.1. Then, the optimal state*

[16]See Helmes et al. (2013), Table 1, for the case where ψ is a power function.

process evolves according to the formula

$$x^\star_\infty(t) = B^{-1}\left(e^{-\gamma r t} B(x_0)\right). \tag{3.59}$$

The optimal open-loop controls are given by

$$p^\star_\infty(t) = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{\gamma r}{\eta} B(x_0)\right)^{-\frac{1}{\gamma}} e^{rt}\psi(x^\star_\infty(t))^{\frac{1}{\varepsilon-1}}, \tag{3.60}$$

and

$$w^\star_\infty(t) = \left[\frac{\Delta}{1-\Delta} r \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}} B(x_0)^{\frac{\gamma-1}{\gamma}} e^{-(\gamma-1)rt}\right]^{\frac{1}{a}}. \tag{3.61}$$

The optimal controls in feedback form are

$$\hat{p}_\infty(x) = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{\gamma r}{\eta} B(x)\right)^{-\frac{1}{\gamma}} \psi(x)^{\frac{1}{\varepsilon-1}}, \tag{3.62}$$

and

$$\hat{w}_\infty(x) = \left[\frac{\Delta}{1-\Delta} r \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}} B(x)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{a}}. \tag{3.63}$$

The sales rates (in open-loop form and in closed-loop form) associated with the optimal controls are given by

$$\lambda^\star_\infty(t) = \gamma r \frac{B(x_0)}{B'(x^\star(t))} \quad \text{and} \quad \hat{\lambda}_\infty(x) = \gamma r \frac{B(x)}{B'(x)}. \tag{3.64}$$

The optimal continuation value and the value function are

$$V_\infty(t) = \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}} B(x_0)^{\frac{\gamma-1}{\gamma}} e^{-(\gamma-1)rt} \quad \text{and} \quad \hat{V}_\infty(x) = \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}} B(x)^{\frac{\gamma-1}{\gamma}}. \tag{3.65}$$

The infinite horizon problem is of special interest as it (approximately) models the situation of a product or industry life cycle. Prominent classical examples include, e.g., the market of black-white TV and analog photography. The finite horizon problem reflects the situation when a product is only available for a certain (and predefined) time and is then superseded by a new model. For example, in the automobile industry this is the situation when a new model of a particular car is introduced; in the movie industry, the horizon is the (fixed) time between sequels of a successful film series. In Chapter 4, we address the problem how to choose the cycle length and the capacity such that the present value of the total profit is maximized. Formula (3.65) in Corollary 3.2.2 yields (at least) an upper bound on the value of the (optimal) capacity x if the sales horizon (cycle length) is very large.

We conclude this section by some remarks concerning the results obtained so far. The detailed analysis and illustration of the results is given in Section 3.3, where we consider a particular example of a system function ψ, the von Bertalanffy adoption rate. We also refer to Helmes et al. (2013) for other examples and particular model settings, specifically, the analysis of the controlled Bass model.

Remark 3.2.3 *Assume all hypotheses which underlie Theorem 3.2.1 and Theorem 3.2.2 hold. The following facts are important.*

- *Optimal prices are dynamic, except for the special case when $\psi(x) \equiv const$ and $r(t) \equiv 0$.*
- *Optimal discounted prices $e^{-R(t)}p^\star(t)$ decrease (increase) over time if ψ increases (decreases) monotonically in x. If ψ monotonically decreases in x, it is optimal to increase the nominal prices $p^\star(t)$ over time; a market penetration strategy is optimal.*
- *The arrival intensity $\mu(t)$ determines the optimal price level via the time-to-go potential. However, fluctuations of the μ function are not reflected in the dynamic of the price process.*
- *It is optimal to set the advertising spending at time t proportional to the value of $\eta(t)$, i.e., fluctuations of the arrival intensity are reflected by the advertising spending.*
- *If the arrival intensity $\mu(t)$ is multiplied by a factor $\alpha > 0$, the expressions for the optimal advertising effort, the continuation value and the value function have to be multiplied by the factor $\alpha^{\frac{1}{\gamma}}$.*
- *If $\mu(t) \equiv \mu$ and $r(t) \equiv 0$, positive respectively, then the optimal advertising rate is constant, decreasing respectively, over time.*

3.3 Controlled von Bertalanffy Models

We now consider the system function ψ to be of the von Bertalanffy type, see (3.3). In a von Bertalanffy model, the innovation channel (the external influence) is shut down. Thus, in the uncontrolled von Bertalanffy model, the initial value must be different from one since otherwise no adoptions occur and the state process remains at its initial value, cf. Section 3.1. In the *controlled* von Bertalanffy model, price and advertising decisions (and the arrival intensity) influence the adoption rate. The solution formula of $x^\star$ for general ψ functions, cf. Theorem 3.2.2, implies that the optimal process drifts away from its initial state x_0 even though $x_0 = 1$. The reason for this behavior is that if x is close to 1 (or 0), $\psi_{vB}(x) = \phi_{vB}(1-y)$ tends to zero such that the optimal price tends to 0: in formula (3.24) for the optimal price the term $\psi(x^\star(t))^{(1/(\varepsilon-1))}$ appears in the denominator, $\varepsilon > 1$. Hence, the associated sales rate becomes very large and tends to $+\infty$. Note, if $x_0 < 1$, the von Bertalanffy system function is strictly positive on $(0, x_0]$, and the expressions for the optimal control and associated sales rate can also be evaluated at $t = 0$.

The von Bertalanffy model lacks the external influence which has to be compensated by *zero* prices if $x_0 = 1$. It is common practice that the producer of a new product provides some copies of its product for free to magazines or test buyers to essentially create such an external influence. An alternative might be to raffle off a handful of items as part of a promotion campaign taking place before the actual sales period begins.[17] Hence, one can think of x_0 being relatively *close* to 1, e.g., $x_0 = 0.99$, the value 0.99 indicates that still 99 percent of the market needs to be taken. In equation (3.3), the quantity y is the actual fractional market share. Since we consider the state $x(t)$ to represent the untapped (fractional) market share at time t, the system function of the von Bertalanffy adoption model is given by, $0 \leqslant x \leqslant 1$,

$$\psi(x) = \psi_{vB}(x) = \phi_{vB}(1-y) = \frac{\Gamma}{1-\theta}\Big[(1-x)^\theta - (1-x)\Big]. \tag{3.66}$$

Recall, if $\theta = 2$, the uncontrolled model reduces to the Mansfield model and, in case $\theta \to 1$, ψ_{vB} becomes the Gompertz curve $\psi(x) = \Gamma(1-x)\log(1/(1-x))$, cf. Section 3.1. Due to the multiplicative demand structure, the coefficient Γ simply acts as a scaling factor that can be assigned to the function μ. From this point of view one can interpret the coefficient Γ as a (modified) response constant. In what follows, we assume $\Gamma \equiv 1$ and capture influences such as the basic demand level or a response constant by the μ

[17] In the context of Chapter 4, where optimal cycle length and capacity decisions are considered, these associated cost are captured by the fixed setup cost k.

function, see below. The second parameter of ψ, θ, captures the imitation behavior of customers. If the θ value is *small*, the influence $(1-x)^\theta$ of the customers who already adopted the product is relatively strong compared to the case of a *large* θ value.[18] Hence, from a retailer's or producer's point of view a market situation with a small value of θ is favorable as early adopters attract more buyers, the demand increases, higher prices can be charged and profits can be realized earlier. Technically, the benefit from a small θ value can be seen by considering the continuation value $V_{x_0}(t)$, cf. equation (3.50), which is an increasing function of the diffusion potential $B(x) = \gamma/(\gamma-1)\int_0^x \psi(z)^{1/(\varepsilon-1)}dz$. It follows from elementary calculus that the value of $\psi(x)$ strictly decreases in θ for every fixed $x \in (0,1)$. Hence, the value of the diffusion potential decreases in θ too. Thus, ceteris paribus, a larger value of θ goes along with a smaller continuation value.

The value of the optimal advertising rate in the open-loop representation also decreases in θ since this value only depends on the system function via the diffusion potential of the initial inventory and not on the current level of the inventory, Corollary 3.2.1. Thus, the *faster* non-adopters are influenced by adopters, i.e., the larger the θ value, the larger the benefit from additional promotion.[19] The dependence of the optimal price and the sales rate associated with the optimal control on θ is less obvious. For the optimal sales rate it is particularly interesting to ask how integrating control variables into the model influences the structural properties like symmetry and the location of the point of inflection (compared to the uncontrolled von Bertalanffy model, cf. Section 3.1). In the following, we consider specific parameter settings to analyze and illustrate characteristics of the controlled von Bertalanffy model.

First, we assume a finite time horizon ($T < \infty$), a time-homogeneous arrival intensity ($\mu \equiv const$), and no discounting ($r = 0$). In this market situation, it is optimal to advertise at a constant level, see (3.25) and Remark 3.2.3. Panel (a) of Figure 3.3 shows the optimal price paths for different values of θ. If $\theta = 0$, a price skimming strategy is optimal throughout the whole sales period: prices start at their highest level and decrease continuously. This property is not surprising since, in this particular case, $\psi(x) = \mathit{\Gamma} x$, and the imitation effect decreases together with the untapped market share over time.[20] Note, in all plots in this section we assume $x_0 = 0.99$. Since $\psi_{vB}(x_0) > 0$,

[18] For example, if $x = 1/2$ and $\theta = 1/10$ the value of the expression $\psi(1/2) = \frac{\mathit{\Gamma}}{1/2}\left[(1/2)^{1/10} - 1/2\right] = \mathit{\Gamma}\left(\sqrt{2}-1\right) \approx 0.41\mathit{\Gamma}$ is larger than $\psi(1/2) = \frac{\mathit{\Gamma}}{-1}\left[(1/2)^2 - 1/2\right] = \mathit{\Gamma}/4$ if $\theta = 2$, the Mansfield model. Naturally, the effect of a particular θ value depends heavily on the value of x.

[19] Actually, one could argue contrarily: due to the higher demand if the θ value is small, the promotional effort can be reduced. In fact, the optimal price policy and the associated sales rates induce that it is profitable to support the fast adoption by additional adverting spending, cf. below.

[20] Effectively, in this case the system function represents an external effect of innovation, cf. the Bass adoption rate if $\mathit{\Omega} = 1$ and $\mathit{\Gamma} = 0$.

this assumption implies that the (optimal) control and associated values of interest can be evaluated at $t = 0$. When θ takes a strictly positive value, it is optimal to apply a price penetration policy at the beginning of the sales period, i.e., the optimal prices increase monotonically up to some point in time $T_{p^\star}$; from that point on price skimming is optimal for the rest of the sales period. The value of $T_{p^\star}$ increases in θ; it equals 0 if $\theta = 0$ and tends to T if θ becomes large, see Proposition 3.3.1 below. In the model

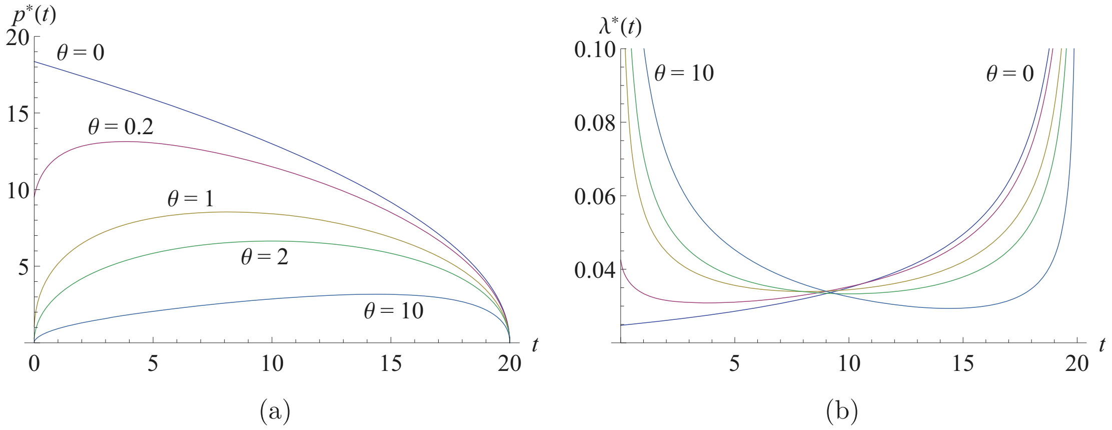


Figure 3.3: Controlled von Bertalanffy model: optimal price trajectories (a) and optimal sales rates (b) as functions of time $t \in [0, T]$ for different θ values and a constant μ function; $x_0 = 0.99, T = 20, \mu(t) = 25, \varepsilon = 2, \Delta = 0.5, r = 0, \Gamma = 1$.

for general ψ functions, cf. Section 3.2, if T is finite and ψ is positive on $(0, x_0)$, an implication of the optimal control is that the whole market potential will be captured at time T, i.e., $x^\star(T) = 0$. In particular, the von Bertalanffy system function fulfills this property, moreover, $\psi_{vB}(0) = 0$ and is strictly concave, see the proof of Proposition 3.3.1 below. The optimal price $p^\star$ tends to zero towards the end of the sales period when the untapped market share values are small. To be precise, the optimal price process is of the form $p^\star(t) = const \cdot \psi_{vB}\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}$, cf. equation (3.24). Thus, the dynamics of $p^\star(t)$ are due to changes in the controlled process $x^\star(t)$. The adoption rates associated with the optimal controls are depicted in panel (b) of Figure 3.3 for the same θ values as in panel (a); colors coincide. Recall, since μ is constant and the interest rate is zero the optimal advertising rate is constant. Thus, the dynamic in the sales process is due to the dynamic of the price process and the influence of the system function only. Consequently, the sales rates $\lambda^\star$ show an inverse behavior compared to the optimal price paths due to the negative exponent $-\varepsilon$ of the price factor in the sales rate. In the extreme case $\theta = 0$ the sales rate monotonically increases since it is optimal to use a price skimming strategy on the interval $[0, T)$. For nonzero values of θ the plots of the controlled adoption rate show a *bathtub* shape. If θ is small, the *bathtub* is skewed to the left; if θ is large, it is skewed to the right. In case $\theta = 2$, the *controlled* Mansfield model, the sales rate - and also the optimal price paths - are *almost*[21] symmetric around $T_{p^\star} \approx T/2$.

One can certainly imagine price paths to follow the behavior as depicted in panel (a) of Figure 3.3 (especially if θ is small). In the context of a new product or technology, however, it is less likely to observe an adoption rate as shown above. Increasing sales rates at the end of the planning horizon are due to declining prices which are set so low at the end of the sales period to acquire the final market shares. In practice, clearance sales are often treated separately from the original planning schedule. For such an application-oriented pricing model in the context of retailing see, for example, Smith (2009). The controlled von Bertalanffy model is not supposed to explain all empirical characteristics of sales rates at any point in time, especially not close to the end of the adoption horizon.[22] Also the rapid decline of the sales rate at the beginning if θ is nonzero, can usually not be expected for a new product. In Figure 3.3, we assume μ to be constant, a fact which is suitable to illustrate the impact of θ on the quantities of interest. In general, however, it is natural to assume a dynamic arrival intensity over a product's life cycle. Figure 3.4 shows optimal price trajectories and the associated sales rates for a parametrized class of μ-functions where μ varies over times. Bemmaor (1994) has proposed to mix special densities to capture a buyer's propensity to buy and he derives a density of first purchase times across all potential buyers. We use slightly modified elements of this class of parametrized density functions as our μ-functions. In particular, we consider a shifted version of equation (6) in Bemmaor (1994) and assume

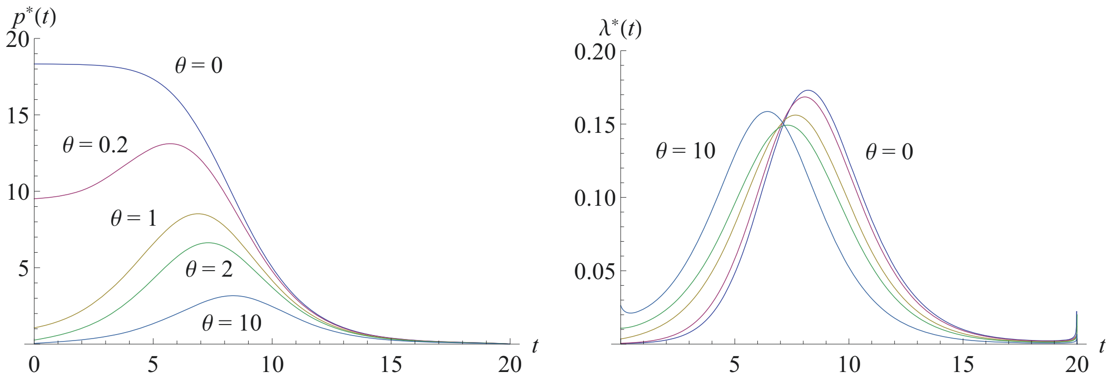
$$\mu(t) = \mu_0 \left(\mu_1 + \frac{\mu_b (1 + \mu_\beta) e^{-\mu_b (t + \mu_s)}}{\left(1 + \mu_\beta e^{-\mu_b (t + \mu_s)}\right)^2} \right), \tag{3.67}$$

where the subscripts b and β correspond to the parameters used by Bemmaor. We introduce the additional parameters μ_s (shift), μ_1 (minimum demand) and the scaling parameter μ_0 in order to fit the μ function to our needs. Note, in this formulation the parameter μ_0 captures the response constant *and* general level effects. In Figure 3.4, we choose the parameter values in such way that the time-to-go potential $A^{(0)}(0)$ is approximately equal to the potential of the constant μ scenario considered in Figure 3.3. In other words: the overall buying interest is similar in both scenarios but differs in the

[21] Due to the initial value $x_0 = 0.99$ the symmetry is not perfect.

[22] A possibility to adopt the model at hand for real applications is to assume a *large* value of T and to consider the sales horizon only until some point in time before T. The remaining sales period can then be modeled separately, e.g., by a clearance pricing model.

concrete time pattern.[23] The pattern of the optimal price process remains the same as in



(a) (b)

Figure 3.4: Controlled von Bertalanffy model: optimal price controls (a) and optimal sales rates (b) at time $t \in [0, T]$ for different θ values if μ follows (3.67); $x_0 = 0.99, T = 20, \mu_0 = 280, \mu_1 = 0.001, \mu_b = 0.6, \mu_\beta = 4, \mu_s = -5, \varepsilon = 2, \Delta = 0.5, r = 0, \Gamma = 1$.

the time-homogeneous setting: if $\theta > 0$, prices increase until $T_{p^\star}$ (penetration strategy) and then decrease monotonically (skimming strategy). However, in contrast to the case where μ is constant, the price drop is more dramatic, especially if θ is *small*. If θ tends to zero, optimal prices are more or less constant at the beginning of the sales period and then show the typical decline.

Similar to the case when μ is constant, the optimal price level decreases in θ - the less persistent the word of mouth effect is, the more customers have to be *convinced* by lower prices. If $\theta = 2$ (the Mansfield model), the optimal price trajectory is still (almost) symmetric around $T_{p^\star}$. However, compared to the scenario when μ is constant, the optimal price reaches its peak earlier. Moreover, $T_{p^\star}$ coincides with the point in time when the arrival intensity peaks. In the context of adoption processes, the adoption rate associated with the optimal price and advertising control[24], see panel (b) of Figure 3.4, shows a pattern that looks more *familiar* than the rates depicted in panel (b) of Figure 3.3. The rate of adoption starts at a relatively low level, and then livens up following the increase in the arrival rate $\mu(t)$, and is further amplified by the word of mouth effect. If $\theta = 2$, the point of inflection coincides with the point in time where the μ function peaks (around 7.3). The point of inflection is shifted to the left if $\theta > 2$, and it is shifted

[23] We refrain from presenting a plot of the μ function here as its shape essentially follows the sales rate when $\theta = 2$ (green line) in panel (b) of Figure 3.4. The μ function takes the value $\mu(0) \approx 2.8$, peaks at 7.3 where it takes the approximate value of 52.8, and decreases until $\mu(T) \approx 0.4$.

[24] Recall, the optimal advertising control co-moves with the function $\mu(t)$, cf. equation (3.25) and the definition of $\eta(t)$ in Definition 3.2.1.

to right if $\theta < 2$. Whenever θ takes a *large* value (low word of mouth influence), sales are pushed by setting lower prices at the beginning since lower prices stimulate early adoption. If the force of imitation is strong (a small θ value, $\theta \ll 2$), higher prices can be charged. This policy is justified by the fact that many non-adopters want to emulate the customers that already purchased the product.

Often - especially from a consumer's point of view - it is of particular interest to know when prices peak. We show that in the undiscounted case this point in time $T_{p^\star}$ is unique and can be characterized in terms of the θ value and the value of the untapped market share.

Proposition 3.3.1 *For a controlled von Bertalanffy model with finite time T, assume $\Gamma > 0, \theta \geqslant 0, 0 < x_0 < 1, 0 \leqslant \delta < a, \varepsilon > 1$, and $r(t) \equiv 0$. Then, there exists a unique point $T_{p^\star} \in [0, T]$ at which the optimal price process (3.24) attains its maximum:*

(i) If $\theta = 0$, then $T_{p^\star} = 0$.

(ii) If $\theta = 1$, then

(a) if $x_0 \leqslant (1 - 1/e) \approx 0.63$, then $T_{p^\star} = 0$,

(b) if $x_0 > (1 - 1/e)$, then $T_{p^\star}$ satisfies

$$x^\star \left(T_{p^\star}\right) = 1 - 1/e. \tag{3.68}$$

(iii) If $\theta > 0, \theta \neq 1$, then

(a) if $x_0 \leqslant 1 - (1/\theta)^{\frac{1}{\theta-1}}$, then $T_{p^\star} = 0$,

(b) if $x_0 > 1 - (1/\theta)^{\frac{1}{\theta-1}}$, then $T_{p^\star}$ satisfies

$$x^\star \left(T_{p^\star}\right) = 1 - (1/\theta)^{\frac{1}{\theta-1}}. \tag{3.69}$$

Proof. In the von Bertalanffy model, if $r(t) \equiv R(t) \equiv 0$, it follows from Theorem 3.2.1 that the optimal price process is of the form $p^\star(t) = const \cdot \psi_{vB}\left(x^\star(t)\right)^{\frac{1}{\varepsilon-1}}$. Since $\varepsilon > 1$, the function $\zeta \mapsto \zeta^{\frac{1}{\varepsilon-1}}, \zeta \geqslant 0$, is strictly increasing. Hence, $p^\star(t)$ is maximized if $\psi_{vB}\left(x^\star(t)\right)$ is maximized.

First, we verify case (iii). Let $\theta > 0, \theta \neq 1$, and let $h(x) := \psi_{vB}(x)$, $x \in [0, 1]$, i.e.,

$$h(x) = \frac{\Gamma}{1-\theta}\Big[(1-x)^\theta - (1-x)\Big]. \tag{3.70}$$

The function h is continuous on the unit interval and strictly positive on $(0, 1)$; $h(0) = h(1) = 0$. Moreover, since $\theta > 0$, the function h is strictly concave on $[0, 1)$; simply compute the second derivative of h: for any x, $0 \leqslant x < 1$,

$$h''(x) = -\Gamma\theta(1-x)^{\theta-2} < 0.$$

Hence, $h(x)$ attains its maximum at a unique point x_{vB}; elementary calculus shows that

$$x_{vB} = 1 - (1/\theta)^{\frac{1}{\theta-1}}. \tag{3.71}$$

On the interval $[0, T]$, the optimal path $x^\star$ strictly decreases from $x^\star(0) = x_0$ to $x^\star(T) = 0$. Thus, if $x_{vB} \geqslant x_0$, then $\psi_{vB}(x)$ attains its maximum at x_0 and the optimal price peaks when $x^\star(t) = x_0$. Hence, $T_{p^\star} = 0$, and this proves statement (a) of part (iii).

If $x_{vB} \in (0, x_0)$, then $\psi_{vB}(x)$ is maximized at x_{vB}, and $p^\star(t)$ is maximized when $x^\star(t) = x_{vB}$. Thus, $T_{p^\star}$ must satisfy $x^\star(T_{p^\star}) = x_{vB}$, and (3.69) follows.

If $\theta = 0$ (case (i)), then h reduces to $h(x) = \Gamma x$, a strictly increasing function. Thus, ψ_{vB} is maximized at the largest feasible value $x_0 = x^\star(0)$. Hence, $p^\star(t)$ is maximized at $T_{p^\star} = 0$, which proves (i).

In order to prove (ii), notice that the function $h(x)$, x fixed, converges to the Gompertz function $h_1(x) = \Gamma(1-x)\log(1/(1-x))$, $0 \leqslant x < 1$, if $\theta \to 1$. As a function of x, elementary calculus shows that h_1 is strictly concave on the unit interval and attains its maximum value at $x_{vB1} = 1 - 1/e \approx 0.63$. Again, if $x_0 > x_{vB1}$, the function ψ_{vB} attains its maximum in the interior of the interval $[0, x_0]$ at x_{vB1}, and $p^\star$ is maximized at $T_{p^\star}$, where $T_{p^\star}$ satisfies $x^\star(T_{p^\star}) = x_{vB1}$. If $x_0 \leqslant x_{vB1}$, the function $h_1(x)$ is strictly increasing on the interval $[0, x_0]$ and h_1 attains its maximum value at x_0. Hence, ψ_{vB} attains its maximum at $x^\star(0) = x_0$. Thus, the maximum of $p^\star$ is attained at $T_{p^\star} = 0$. ♦

An implication of Proposition 3.3.1 is the following one: if x_0 is small relative to the expression on the right-hand side of $(iii)(a)$, a price skimming strategy is optimal throughout the sales period. Price skimming is also optimal if the word of mouth effect depends linearly on the value of the untapped market share ($\theta = 0$); this result can also be found in Section 4 of Helmes et al. (2013) for the case $\psi(x) = x^b$ and $b = 1$. Apart from these special cases, the controlled von Bertalanffy model entails that it is optimal to start with a price penetration strategy; once the price peaks it is optimal to use a price skimming strategy. Note, the optimal price at time t does not depend on the specific value of $\mu(t)$ but only on the time-to-go potential $A^{(0)}(0, T)$. Even if the arrival rate is assumed to oscillate, for example due to seasonal patterns, the *first-penetrate-then-skim* pricing strategy is optimal in the controlled von Bertalanffy model. The value of $T_{p^\star}$ is determined by equations (3.68) and (3.69) and thus naturally depends on all parameters that influence the controlled adoption process. However, the *value* of the untapped market share at time $T_{p^\star}$ is uniquely characterized by the value of θ. Assuming $x_0 = 1$ and $\theta = 2$, then the optimal price trajectory attains its maximal value when exactly one half of the total market potential has been captured. If $\theta < 2$, then less than half of the market share has been acquired when prices reach their maximum. If $\theta > 2$, more than fifty percent of the total market potential have been tapped if the prices peak. If θ is *very large*, then a penetration strategy is optimal for most of the sales horizon. For instance, if $\theta = 20$, then $x^\star(T_{p^\star}) \approx 0.15$, i.e., optimal prices reach the peak after approximately 85 percent of the total market has been captured.

In general, the characteristics of the function ψ *infuse* the price and the adoption rate of the model. That is why the controlled Bass model is not able to fit situations where prices peak after more than half of the market share has been captured.[25] In contrast, the class of von Bertalanffy models is able to fit more general price processes and sales processes. This feature is particularly important when (theoretical) models are fitted to empirical data. In applications, the specification of the model parameters is the ultimate challenge. For a specific class of products there is often a general agreement on the values Γ and θ based on intuition and/or experience. Very important but especially difficult is to estimate the function μ. The analysis is further complicated by the fact that reliable estimates of the elasticities (δ and thus Δ, and ε) are difficult to obtain. In practical applications, the management will find it useful to first assume one or more specific scenarios and then to update the model and its parameters by more reliable data and estimates which become available over time.[26]

[25]An analysis along the lines of Proposition 3.3.1 for $h(x) = \psi_{Bass} = \Omega + \Gamma x(1-x)$, $\Omega, \Gamma > 0$, shows that in the controlled Bass model the corresponding $T_{p^\star}$ value satisfies $x^\star(T_{p^\star}) = 0.5 + \Omega/(2\Gamma) > 0.5$.

[26]In this context, one can also make use of the feedback controls, cf. Theorem 3.2.2, or adjust values such as x_0 and parameters accordingly.

4 Maximizing Long-Term Profit

4.1 Introduction

In Chapter 2 and Chapter 3, we consider *one-cycle* control problems with a given sales period of length T. However, inventory management is a matter of recurring decisions. A very basic - but probably the most famous - approach to inventory control is the *economic order quantity* (EOQ) method which, presumably, was first analyzed by Harris (1913). Given an (exogenous) constant demand rate λ the decision maker has to choose an inventory capacity (or lot size) $x_0 > 0$. Having chosen x_0 and given λ, the inventory will deplete over a cycle of length $\tau = x_0/\lambda$. Independent of the size of x_0, an order cost $k > 0$ has to be paid once per cycle; holding costs accrue at a constant rate ℓ per unit and unit of time. The objective of the decision maker is to choose an inventory level x_0 such that the average cost (per time unit) is minimized.[1] The resulting EOQ x_{HW} is given by

$$x_{HW} = \sqrt{\frac{2k}{\ell}\lambda}.$$

The corresponding optimal cycle length equals $\tau_{HW} = x_{HW}/\lambda = \sqrt{2k/(\ell\lambda)}$, and the minimized cost per time unit amounts to $C_{HW} = \sqrt{2k\ell\lambda}$. Since also Wilson (1934) has his merits in the development and analysis of this model, this model is sometimes called the *Harris-Wilson* EOQ or *Harris-Wilson* model, and this explains the $'HW'$ subscript.[2] Although the assumptions of the *Harris-Wilson* setting are rarely met in reality, the results are widely used in practice. Reasons are that the formulas are easy to compute and to implement, and that the results show a remarkable robustness with respect to parameter misspecifications and uncertainties. The robustness property is due to the square root function in all the expressions. We will seize this idea of a simple but robust formula in Section 4.3.3 and derive an *endogenized Harris-Wilson* formula

[1] Naturally, instead of searching for the best value of x_0, one can search for the optimal cycle length. If τ_0 is the optimal cycle length, then the optimal lot size equals $x_0 = \lambda\tau_0$. Both approaches are equivalent.

[2] Independent of Harris and Wilson, Andler (1929) developed this basic method of inventory management in the context of production and optimal batch sizes, see Krieg (2005) for a historic and economic classification of the contribution by Andler.

that - in contrast to the classic EOQ model where the constant demand rate is given - endogenizes the demand rate.

Starting from the beginning of the last century many modifications and extensions of the basic model have been discussed in the (inventory) literature: single- and multi-echelon systems, deterministic and stochastic demand models, continuous and periodic review systems, continuous and integer lot sizes, independent and coordinated replenishments, the implication of leadtimes, backorders, discounts, perishability, and many other aspects. The textbooks by Axsäter (2006) and Snyder and Shen (2011) give detailed reviews of the current state of inventory control. The authors describe many theoretical models, and suggest practical methods and heuristics. Classical textbook references in inventory control are, for instance, Silver et al. (1998) and Zipkin (2000).

Usually, optimal inventory control is concerned about minimizing costs assuming that the (deterministic or stochastic) demand is exogenously given. But as Axsäter (2006) nicely points out in his introduction, pp. 1, inventories "... cannot be decoupled from other functions, for example purchasing, production, and marketing. As a matter of fact, the objective of inventory control is often to balance conflicting goals. One goal is, of course, to keep stock levels down to make cash available for other purposes. The purchasing manager may wish to order other large batches to get volume discounts. The production manager similarly wants long production runs to avoid time-consuming setups. He also prefers to have a large raw material inventory to avoid stops in production due to missing materials. The marketing manager would like to have a high stock of finished goods to be able to provide customers a high service level." Naturally, the optimal batch size or inventory level depends on the demand rate. But if the monopolist is in the position to choose the prices which influence the demand, too, then decisions of setting optimal prices and choosing an optimal batch size interact. If marketing and production aspects enter the decision process, the complexity of the optimization problem increases further. As soon as pricing is involved, one is typically interested in maximizing profits rather than minimizing costs which distinguishes the pure inventory management problems from revenue and supply chain management in general.

In management science, simultaneous pricing and inventory decision making plays an important role from the 1950s on. Whitin (1955) was among the first who incorporated pricing decisions into an inventory problem and pioneered this field of research. Reviewing all the results related to joint pricing-inventory would go beyond the scope of this work. We thus reference only the research that is closely related to our approach and refer to review articles by Eliashberg and Steinberg (1991), Elmaghraby and Keskinocak (2003), Chan et al. (2004), Yano and M. (2004), or Chen and Simchi-Levi (2012) for an overview of the last decades on this research area.

The work by Pekelman (1974) laid the foundation for many papers on pricing and inventory decision making using control theory. Pekelman applies control theory techniques in order to simultaneously determine time-dependent price and production rates over a continuous finite time horizon. He assumes a linear time- and price-dependent demand rate and a strictly convex production cost function. The state of the system is the inventory level which is assumed to be nonnegative and is continuously refilled via a production process with adjustable rate. He characterizes the optimal price-production decision and shows that it depends on the sum of the adjoint variable related to the inventory process and the Lagrange multiplier associated with the state constraint, cf. the proof of Theorem 3.2.1 in the previous chapter. Feichtinger and Hartl (1985) extend the model by Pekelman to the nonlinear demand case and to account for shortage costs. Li (1988) considers a stochastic setting, where the cumulative demand and the cumulative production are governed by two Poisson counting processes with random intensities parameterized by production capacity and price respectively. The textbook by Sethi and Thompson (2000) includes a comprehensive survey on pricing-inventory models solved by control theory techniques up to the late 20th century.

Cohen (1977) is one of the first researchers who incorporates a deterioration effect in the joint pricing and inventory decision making process. He considers a modified EOQ model[3] with constant inventory, production, and setup costs. The deterministic sales rate is a function of price; in addition to sales, the inventory decays at a constant exponential rate. The objective is to maximize the average profit per time unit by choosing a stationary price and a (periodic) cycle length. The inventory level at the beginning of each cycle, the initial inventory, is then implicitly defined by the associated state process and the cycle length - an assumption that is similar to our assumption in Chapter 2 and that enabled us to formulate the problem in terms of a cost function $c(t)$, cf. (2.7) and comments there. Cohen characterizes the optimal solution in form of necessary and sufficient conditions and illustrates his results for the case of a linear demand function. In a second step, he extends the models by backlogging and deduces economic implications.

Rajan et al. (1992) generalize the model of Cohen (1977) by introducing *dynamic* pricing; in addition, they allow for time-dependent demand rates and deterioration rates. They consider the holding costs to be constant over time and disregard any discounting effects. This is a special case of our general dynamic advertising and dynamic pricing

[3]Here and subsequently, the term *modified EOQ model* indicates that the objective is to maximize the average (net or total) profit per unit of time.

model introduced in Chapter 2 when $\delta(t) = 0, \ell(t) = \ell$, and $r(t) = 0$. Rajan et al. (1992) first solve the dynamic pricing problem pointwise for every $t \in [0, T]$ to obtain the optimal profit margin $\nu_R(t)$, cf. Proposition 2.2.2; recall, $\nu_R(t)$ does not depend on the cycle length T. In a second step, the objective is then to maximize

$$\frac{1}{T}\left(\int_0^T \nu_R(t)dt - k\right) \tag{4.1}$$

with respect to the cycle length $T > 0$ in order to obtain an optimal cycle length T^*. Rajan and co-authors impose restrictions on the (optimal) profit function ν_R that guarantee the existence and uniqueness of $T^* > 0$. We generalize the inventory model of Rajan et al. (1992) by incorporating dynamic advertising decisions and time-dependent holding costs as well as allowing for discounting.

While the coordination of pricing and inventory decisions is actually a vibrant research area for more than sixty years, the combination of advertising, pricing, and inventory considerations is a relatively neglected field of research. Urban (1992) is one of the first authors who combines pricing and advertising in the classical lot size problem. The production costs are assumed to be a linear function of the production time and Urban considers a time-homogeneous ($\mu(t) \equiv \mu$) version of the demand function (2.4). The goal is to simultaneously determine a (fixed) price, a constant advertising expenditure rate, and a lot size that maximize the average profit per time unit. The optimal price is determined by a constant markup on the production costs; the production time, i.e., the cycle length, is implicitly defined by the lot size. Urban considers constant holding and setup costs and incorporates shortages and the possibility of defective items. He makes use of separable programming techniques[4] to simultaneously determine all quantities of interest. For special cases, e.g., constant production costs and particular demand patterns, closed-form solutions are derived. Lee and Kim (1993) consider the price to be fixed but allow the unit production cost to be a power form expression of the demand. They consider a model where the three quantities of interest (price, advertising effort, and capacity) are determined simultaneously; Lee and Kim call this the *full integration* model. In what they call *partial integration* model, they separate the pricing-marketing decision from the lot size problem. The authors make use of geometric programming to solve the nonlinear problems.[5] Lee and Kim use marginal analysis to compare the

[4]By making several substitutions Urban modifies the objective function to be a separable function of the decision variables and then determines the range of values that the variables can realize, see Appendix A in the original reference.

[5]For a classic textbook on geometric programming see, for example, Duffin et al. (1967), and see Lee (1993) for a geometric programming algorithm in the context of pricing and inventory considerations.

results of both models, the full integration model and the partial integration model, with respect to managerial implications. Subsequent works consider various extensions and modifications. For example, Ulusoy and Yazgac (1995) extend the model of Urban (1992) by taking multi-product and multi-period aspects into account; Mondal et al. (2007) incorporate transportation costs (if the inventory is replenished); S. J. Sadjadi and Yousefli (2010) consider the demand as a function of price and marketing expenditure with *fuzzy*[6] parameters. Shah et al. (2013) extend the model of S. J. Sadjadi and Yousefli (2010) by considering non-instantaneous deterioration effects in the inventory and time-dependent holding costs.

Although all of these articles consider particular features and challenges of inventory management, to our knowledge, the relationship between dynamic pricing, dynamic advertising and capacity decision models has not been fully analyzed so far. Moreover, we will allow the interest rate to influence future profits. The vast majority of the literature on (multi-period) inventory control disregards inflation effects, although discounting has always been an essential ingredient in revenue management. In their article *Recent trends in modeling of deteriorating inventory*, Goyal and Giri (2001) note on page 11 that the "... basic assumption in the classical EOQ model is that all the cost components associated with the inventory system remain constant over time. Before 1970, the effect of inflation was not considered for analyzing inventory systems perhaps because of the belief that inflation would not influence the inventory policy to any significant degree. However, the situation changed radically in the 1970s when the actual inflation rate of the Western countries shot up to be in the range $8-20\%$ and as a matter of fact the usual EOQ solution required necessary modifications." Goyal and Giri credit Buzacott (1975) to be the first to account for such modifications; further contributions are assigned to Bose et al. (1995), Su et al. (1996), and Sarker et al. (2000) to name just a few examples.[7] Note, all these papers are only related to inventory and supply chain management and include no pricing decisions nor marketing considerations.

For the one-cycle control problems, so far, we only considered the costs of advertising and, in Chapter 2, the costs that are associated with the sale of a unit, the purchasing and the inventory costs. From now on, we will also consider the (fixed) setup or order cost. The parameter $k > 0$ specifies this cost that occurs once per cycle. In practical applications, the setup costs can refer to many aspects: the rent of the sales area and of the storage area, (fixed) labor costs, insurance and maintenance costs of a shop or of a vehicle fleet, franchise fees, etc. All these costs arise from running the business and are

[6] See S. J. Sadjadi and Yousefli (2010) and references therein for their definition of fuzzy parameters in the context of pricing and marketing planning models.

[7] See Chapter 9 in Goyal and Giri (2001).

independent of the number of goods produced, purchased, stored, or sold: whether one item or one thousand items are sold, the rent has to be paid. Throughout this chapter, a capacity of zero is equivalent to *running no business*: no expenses are incurred - in particular no order cost k - and sales are nil. Since the profit from the *no business strategy* is zero, it is an option the decision maker has if market conditions are bad, e.g., if the sales potential is too small to cover the fixed cost k, see below.

We let $\tau > 0$ be the variable cycle length. Instead of a fixed selling horizon T, the monopolist now faces the problem of choosing the length of N cycles; N is a positive integer or symbolizes infinity. We consider the cases of maximizing the discounted N-period profit (N finite or infinite), and the maximization of the average profit per time unit (N infinite).[8] While the classical approach in inventory management is to minimize the average cost per time unit (or to maximize the average profit per time unit if price considerations are involved), we also consider the problem of maximizing the present value of N inventory cycles when N is a finite integer. This allows us to deal with applications where the monopolist faces a fixed number of order decisions, e.g., the marketing of a film series, the retailing of a car series, or the publication of the new version of a book or magazine. Although the concrete number of sequels of a series or movies is seldom predefined (and will often depend on the success of the last model or episode), one can often observe that the time between two editions of a series is constant. For example, one of the best selling series in the history of computer games, the *Civilization* series, appears with a new title every five years starting from 1991.[9] To give a second example from the entertainment industry, between 1977 and 1989 the successful film series *James Bond* appeared with a new episode every two years. But how was this time distance, the cycle length in our terms, chosen? The following framework shall help to analyze this kind of problems; in practical applications, the number N may serve as an initial planning horizon that will be adjusted later.

The notation is consistent with the previous chapters. The following assumptions will be imposed throughout Chapter 4.

Condition 4.1.1

1. *The price elasticity, the advertising elasticity, the advertising cost coefficient, and the interest rate are constant:* $\varepsilon(t) \equiv \varepsilon > 1$, $\delta(t) \equiv \delta \geqslant 0$, $a(t) \equiv a > \delta$, $r(t) \equiv r \geqslant 0$, $t \geqslant 0$.

[8] Generally, we associate the average profit with no discounting. Since in many cases the subsequent analysis remains valid with a positive discount rate, we frequently allow $r > 0$ for the average profit problem, see below.

[9] There has been a shift between *Civilization III* and *Civilization IV* which have been published in 2001 and 2005, respectively. *Civilization V* has been published in 2010.

2. *Let* $\tilde{\mu}(t) > 0$, $\tilde{\ell}(t) \geqslant 0$, *and* $\tilde{q}(t) \geqslant 0$, $t \geqslant 0$, *be functions which are bounded from above. Let* $\tau > 0$ *be arbitrary but fixed. For all* $i \in \mathbb{N}$ *let* $i_\tau := [i\tau, (i+1)\tau)$. *We define* μ, ℓ, *and* q *to be periodic functions of (variable) period length* τ, *i.e.,*

$$\mu(t) = \sum_{i=0}^{\infty} \tilde{\mu}(t - i\tau)\mathbb{1}_{i_\tau}(t), \tag{4.2}$$

$$q(t) = \sum_{i=0}^{\infty} \tilde{q}(t - i\tau)\mathbb{1}_{i_\tau}(t), \tag{4.3}$$

$$\ell(t) = \sum_{i=0}^{\infty} \tilde{\ell}(t - i\tau)\mathbb{1}_{i_\tau}(t). \tag{4.4}$$

If $\tau = \infty$, *let* $\mu(t) \equiv \tilde{\mu}(t), q(t) \equiv \tilde{q}(t)$, *and* $\ell(t) \equiv \tilde{\ell}(t)$ *for all* $t \geqslant 0$.

Across cycles, customers are supposed to behave similarly within each cycle; so do the running costs and the depreciation rate.[10] Whenever we consider the N-cycle problem, we assume an identical length for each cycle since the values of the parameters do not change from cycle to cycle. If the parameters are not periodic functions (that depend on the cycle length), an optimal fixed cycle pattern might not exist. This is due to the fact that it might be better to vary the length of different cycles.

In Section 4.2, we assume the monopolist to apply the optimal pricing-advertising scheme derived in Section 2.2. If so, then profits accumulate at rate $\nu^* > 0$, cf. Theorem 2.2.1. Since the (optimal) profit margin does not depend on the time-to-go, it is independent of the cycle length. Thus, we consider the two-dimensional optimization problem of determining a pair consisting of a cycle length and a capacity value that maximizes the present value of N cycles (see Section 4.2.1), or the average profit per time unit (see Section 4.2.2) assuming that the profit maximizing (price and advertising) control is applied throughout the cycle. One key property in Chapter 2 was that the optimal capacity is implicitly given in terms of the (optimal) sales rate. Thus, we can simplify the two-dimensional optimization problem by first solving for the optimal cycle length; the optimal capacity is then given by (2.7). This way, we reduce the two-dimensional problem to a one-dimensional one. As an alternative, we could first consider the one-dimensional optimization problem with respect to the capacity variable. The optimal cycle length is then implicitly determined. From a practical point of view it seems preferable to first analyze the capacity problem. Usually, entrepreneurs think in terms of (total) volumes of sales and not in terms of lengths of sales periods. However,

[10] Basically, one can also assume the elasticities and even the discount rate to be periodic functions. We abstain from doing so in order not to overburden the notation and to concentrate on - in our view - the really important aspects.

to be consistent with Chapter 2 we choose the cycle length as the quantity of interest.

The model considered in Chapter 2 does not allow for an inventory-dependent demand rate. Therefore, in Section 4.2, we assume that $\psi(x) \equiv 1$. In contrast, the new-product adoption model in Chapter 3 takes the dependence on the state of the system into account. Thus, in Section 4.3, we assume that the optimal control derived in Chapter 3 is applied throughout the cycle. In Chapter 3, the only costs considered are the costs of advertising. The optimal control, see Theorem 3.2.1, maximizes the net present value of the total revenue (minus advertising costs) for a given period and a given capacity; the maximized revenue equals the value function $\hat{V}(0, x_0)$. In Section 4.3, we evaluate the total inventory costs associated with the revenue maximizing control. The total production costs are of the form unit cost times the capacity, and a setup cost $k > 0$ is also considered. The objective is to find an optimal pair, consisting of cycle length and capacity, that maximizes the present value of N cycles (see Section 4.3.1), or to find a pair that maximizes the average profit per time unit (see Section 4.3.2). In both of these sections, we also consider the one-dimensional subproblems when (i) the capacity is fixed and we maximize with respect to the cycle length and when (ii) the cycle length is fixed and we maximize with respect to the capacity. In practical applications, case (i) occurs whenever the total capacity is predefined, e.g., the total number of seats in a concert hall, the number of goods in a retailing store, or the deposit of a natural resource. Then, the (stylized) problem is to determine the optimal cycle length, e.g., the optimal interval between performances of an artist, the selling period for the capacity given, or the extraction time and development of a new resource. In case (ii), the decision maker has to choose the (optimal) capacity while the cycle length is predefined, e.g., the capacity of a stadium for which a football match is scheduled every two weeks or the capacity of an airplane commuting between Berlin and London.[11]

Naturally, since only the advertising costs are considered in the determination of the revenue maximizing controls the solution to the problem in Section 4.3 will be suboptimal compared to the solution derived in Section 4.2 (in the case $\psi(x) \equiv 1$). Nevertheless, in many applied cases the results of Section 4.3 will lead to good decision rules. Due to the complexity of the expressions involved the optimal values can generally only be determined numerically. However, in a very special case, we suggest an *endogenized Harris-Wilson* formula for the optimal cycle length and the optimal capacity, cf. Proposition 4.3.8.

[11] Although in airline ticketing problems the state variable is an integer, a continuous variable is a good approximation.

4.2 Optimal Pricing, Advertising and Inventory Control

A key result of Theorem 2.2.1 in Chapter 2 is the property that the (optimal) profit margin ν^* only depends on the current time t, but does not depend on the length of the sales horizon T. Instead of a fixed and exogenously given T, we now consider the cycle length to be a decision variable; we emphasize this fact by letting τ denote the variable cycle length. Thus,

$$\pi_1^*(\tau) = \int_0^\tau \nu^*(t)dt \tag{4.5}$$

is the maximal present value of an inventory cycle of length τ, see Corollary 2.2.1. Since $\nu^*(t)$ is positive for all $t \geqslant 0$, cf. Theorem 2.2.1, $\pi_1^*(\tau)$ strictly increases in τ. The integral $\pi_1^*(\tau)$, however, may be bounded as a function of τ should $\nu^*(t)$ converge to zero if $t \to \infty$. A crucial assumption that underlies all models and all parameter settings to be considered in Chapter 4 is the following one: the market conditions allow for a cycle profit that defrays at least the fixed cost k. Throughout Chapters 2 and 3 the order cost k has not been considered since it has no influence on the optimal control.[12] In this chapter, however, we define the *minimum cycle length* $T_0 := T_0(k)$ as the first hitting time

$$T_0 := \inf \left\{ T \,\middle|\, \int_0^T \nu^*(t)dt \geqslant k, T \geqslant 0 \right\}, \tag{4.6}$$

where T_0 is set to $+\infty$ if $\int_0^\infty \nu^*(t)dt < k$. If T_0 is infinite, the profit margin is too small to make up for the fixed cycle cost, and it is optimal not to run the business. Naturally, a cycle length chosen by the decision maker must satisfy $\tau \geqslant T_0$ as otherwise the profit (of one cycle) will be negative. Recall, the capacity is implicitly given by the choice of τ and sales rate λ^* associated with the profit maximizing control, cf. (2.7). Hence, the quantities of interest will only depend on the variable cycle length τ, and we face a one-dimensional optimization problem, cf. Section 4.1.

In the next subsection, we will maximize the net present value of the sum of profits of N (identical) periods. To this end, in Proposition 4.2.1, we define auxiliary functions that facilitate our further analysis. We derive conditions that guarantee the existence of an optimal cycle length when N is finite (Theorem 4.2.1) and N is infinite (Theorem 4.2.3) and we characterize the optimal cycle length in terms of the solution of a nonlinear equation. We analyze and illustrate these results by an example and derive structural properties of the optimal solution (Corollary 4.2.1). In Subsection 4.2.2, we

[12] In Chapter 2, the fixed cost implies a lower bound on the parameter T as the profit rate has been maximized pointwise. In Chapter 3, no costs but the advertising costs were considered.

consider the maximization of the average profit per time unit - the *classical* objective in inventory management. Similar to Subsection 4.2.1, we derive conditions that guarantee the existence of an average-optimal cycle length (Theorem 4.2.3) and analyze structural properties; in particular, we examine how advertising effects the optimal cycle length and the optimal average profit (Theorem 4.2.4); see also Corollary 2.2.3.

4.2.1 Maximizing the Present Value of N Cycles

The N-stage problem of a monopolist is to determine optimal cycle lengths τ_i, $i = 1, \ldots, N$, her objective is to maximize the net present value of the total profit. Within each cycle the control is chosen (optimally) according to Theorem 2.2.1. Thus, the expression to be maximized is given by

$$\pi_1^*(\tau_1) - k + e^{-R(\tau_1)}(\pi_1^*(\tau_2) - k) + e^{-R(\tau_1+\tau_2)}(\pi_1^*(\tau_3) - k) + \ldots + e^{-R\left(\sum_{i=1}^{N-1}\tau_i\right)}(\pi_1^*(\tau_N) - k),$$

where $\pi_1^*(\tau) = \int_0^\tau \nu^*(t)dt$ is the optimal net profit of a cycle of length τ excluding the setup cost k. Since the parameter functions show the identical behavior starting from the beginning of each new cycle, cf. Condition 4.1.1, we assume the optimal cycle lengths to be identical for each cycle, i.e., $\tau_1 = \tau_2 = \ldots = \tau_N$. Then, the expression to be maximized with respect to $\tau > 0$ becomes

$$(\pi_1^*(\tau) - k) \sum_{i=0}^{N-1} e^{-R(i\tau)}.$$

Since we assume a constant interest rate, cf. Condition 4.1.1, the sum of discounting factors can be represented by the common geometric series formula.[13] Next, we define a function Υ that depends on this geometric series formula and its derivatives, and we verify properties of Υ.

Proposition 4.2.1 *Let $r, T > 0$ and let N be a positive integer.*

1. *Let $S_N(T)$ denote the present value of an annuity of size one which is paid from today until period $N-1$ and which is periodically discounted by e^{-rT}, i.e.,*

$$S_N(T) = S_N(T; r) := \sum_{i=0}^{N-1} e^{-irT} = \frac{1 - e^{-NrT}}{1 - e^{-rT}}. \tag{4.7}$$

[13] We assume that $r(t)$ is constant. Hence, the cumulative discount factor $R(\tau)$ equals $r\tau$. However, the problem can be analyzed for any periodic discount function. This is also true for the elasticities, which are assumed to be constant in Condition 4.1.1.

Let $\dot{S}_N$, $\ddot{S}_N$ resp., $N \geqslant 2$, denote the first, second resp., derivative of S_N with respect to T, i.e.,

$$\dot{S}_N(T) := \frac{dS_N}{dT}(T) = -r \sum_{i=1}^{N-1} i\, e^{-irT} = -\frac{rS_N(T)}{e^{rT}-1}\left(1 - N\frac{e^{rT}-1}{e^{NrT}-1}\right), \tag{4.8}$$

and

$$\begin{aligned}\ddot{S}_N(T) &:= \frac{d^2 S_N}{dT^2}(N) = r^2 \sum_{i=1}^{N-1} i^2 e^{-irT} \\ &= r^2 S_N(T)\frac{\left(1+e^{rT}\right)\left(e^{NrT}-1\right) - N\left(e^{rT}-1\right)\left[N\left(e^{rT}-1\right)+2\right]}{\left(e^{rT}-1\right)^2\left(e^{NrT}-1\right)}.\end{aligned} \tag{4.9}$$

Then, $S_N(T) > 1$, $S_N(T) < N$, $\dot{S}_N(T) < 0$, and $\ddot{S}_N(T) > 0$, i.e., $S_N(T)$ is a (strictly) monotonically decreasing and (strictly) convex function that is bounded from below by the value one and bounded from above by the value N.

2. *Let $S(T)$ denote the present value of a perpetual annuity of size one, i.e.,*

$$S(T) = S(T;r) := \sum_{i=0}^{\infty} e^{-irT} = \lim_{n\to\infty} S_n(T) = \frac{1}{1-e^{-rT}}. \tag{4.10}$$

Let $\dot{S}$ and $\ddot{S}$ denote the first and second derivative of S with respect to T, respectively, i.e.,

$$\dot{S}(T) := \frac{dS}{dT}(T) = -r \sum_{i=1}^{\infty} i\, e^{-irT} = -\frac{r\,S(T)}{e^{rT}-1}, \tag{4.11}$$

and

$$\ddot{S}(T) := \frac{d^2 S}{dT^2}(T) = r^2 \sum_{i=1}^{\infty} i^2 e^{-irT} = r^2 \frac{e^{rT}+1}{\left(e^{rT}-1\right)^2} S(T). \tag{4.12}$$

Then, $S(T) > 1$, $\dot{S}(T) < 0$, and $\ddot{S}(T) > 0$, i.e., $S(T)$ inherits the structural properties of $S_N(T)$.

3. *Let $N \geqslant 2$. We define, $\Upsilon_N(T) = \Upsilon_N(T;r)$,*

$$\Upsilon_N(T) := \frac{\ddot{S}_N(T)}{\dot{S}_N(T)} - \frac{\dot{S}_N(T)}{S_N(T)} = -\frac{2re^{(N+1)rT}\left[\cosh(NrT) - 1 - N^2\left(\cosh(rT)-1\right)\right]}{\left(e^{rT}-1\right)\left(e^{NrT}-1\right)\left[e^{NrT}-1-N\left(e^{rT}-1\right)\right]}, \tag{4.13}$$

and

$$\Upsilon(T) := \lim_{n\to\infty} \Upsilon_n(T) = -\frac{r}{1-e^{-rT}} = -rS(T). \tag{4.14}$$

Then,

a) $\Upsilon_N(T)$ *and* $\Upsilon(T)$ *are negative numbers,*

b) $\Upsilon_N(T) > \Upsilon_{N+1}(T)$*,*

c) $-\frac{r}{1+e^{-rT}} \geqslant \Upsilon_N(T) > -\frac{r}{1-e^{-rT}}$*,*

d) $\lim_{T\to\infty} \Upsilon_N(T) = -r$*,*

e) $\lim_{r\to 0} \Upsilon_N(T) = \lim_{r\to 0} \Upsilon_N(T;r) = 0$*.*

Proof. The formula for $S_N(T)$, $S(T)$ respectively, is the well-known formula of a finite, infinite respectively, geometric series; the formulas of both derivatives are obvious. To see that the first derivatives are negative and that the second derivatives are positive use the representation of $\dot{S}_N(T)$ as a finite, see (4.8), and the representation (4.11) as an infinite series as well as (4.9) and (4.12). Apparently, $\lim_{T\to\infty} S_N(T) = 1$, and, by l'Hôpital's rule, $\lim_{T\to 0} S_N(T) = \lim_{T\to 0} \frac{1-e^{-NrT}}{1-e^{-rT}} = \lim_{T\to 0} \frac{Nre^{-NrT}}{re^{-rT}} = N$. Thus, S_N is bounded by 1 and N.

The derivation of the formula for $\Upsilon_N(T)$ requires more effort. After some lengthy but elementary calculations, or with the help of a proper software package[14], one can see that

$$\frac{\ddot{S}_N(T)}{\dot{S}_N(T)} - \frac{\dot{S}_N(T)}{S_N(T)} = -\frac{re^{(N+1)rT}\left[e^{NrT} + e^{-NrT} - N^2\left(e^{rT} + e^{-rT}\right)) + 2\left(N^2 - 1\right)\right]}{\left(e^{rT} - 1\right)\left(e^{NrT} - 1\right)\left[e^{NrT} - 1 - N\left(e^{rT} - 1\right)\right]}.$$

Using the definition of the cosh function, $\cosh(x) := \frac{1}{2}\left(e^x + e^{-x}\right)$, one obtains expression (4.13). Formula (4.14), the limiting case of (4.13) if $N \to \infty$, is best proved by exploiting the derivatives $\dot{S}$ and $\ddot{S}$. Next, we prove the statements $(a) - (e)$.

(a) Evidently, $\Upsilon(T) < 0$. In order to show that $\Upsilon_N(T) < 0$, we make use of the power series representation of the exponential function, $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, and the series representation of the cosh function, $\cosh(x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!}$. The numerator of $\Upsilon_N(T)$ is

[14]We have used *Mathematica 9* by Wolfram Research, Inc. (2012).

positive since $re^{(N+1)rT} > 0$ and

$$\begin{aligned}\cosh(NrT) - 1 - N^2\left(\cosh(rT) - 1\right) &= \left(1 + \frac{(NrT)^2}{2} + \sum_{i=2}^{\infty} \frac{(NrT)^{2i}}{(2i)!}\right) - 1 \\ &\quad - N^2\left[\left(1 + \frac{(rT)^2}{2} + \sum_{i=2}^{\infty} \frac{(rT)^{2i}}{(2i)!}\right) - 1\right] \\ &= \sum_{i=2}^{\infty} \frac{(NrT)^{2i}}{(2i)!} - N^2 \sum_{i=2}^{\infty} \frac{(rT)^{2i}}{(2i)!} \\ &= \sum_{i=2}^{\infty} \left(N^{2i} - N^2\right) \frac{(rT)^{2i}}{(2i)!} \\ &> 0.\end{aligned}$$

Similarly, the denominator of $\Upsilon_N(T)$ is positive since $\left(e^{rT} - 1\right)\left(e^{NrT} - 1\right) > 0$ and

$$\begin{aligned}e^{NrT} - 1 - N\left(e^{rT} - 1\right) &= \left(1 + NrT + \sum_{i=2}^{\infty} \frac{(NrT)^i}{i!}\right) - 1 \\ &\quad - N\left[\left(1 + rT + \sum_{i=2}^{\infty} \frac{(rT)^i}{i!}\right) - 1\right] \\ &= \sum_{i=2}^{\infty} \frac{(NrT)^i}{i!} - N \sum_{i=2}^{\infty} \frac{(rT)^i}{i!} \\ &= \sum_{i=2}^{\infty} \left(N^i - N\right) \frac{(rT)^i}{i!} \\ &> 0.\end{aligned}$$

Hence, $\Upsilon_N(T) < 0$; note the minus sign in (4.13).

(b) To see that $\Upsilon_N(T)$ is monotone increasing in N, we rewrite $\Upsilon_N(T)$ using the power series expressions of the numerator and the denominator, see above. Obviously,

$$\begin{aligned}\Upsilon_N(T) &= -\frac{2re^{(N+1)rT}}{\left(e^{rT} - 1\right)\left(e^{NrT} - 1\right)} \frac{\sum_{i=2}^{\infty} \left(N^{2i} - N^2\right) \frac{(rT)^{2i}}{(2i)!}}{\sum_{i=2}^{\infty} \left(N^i - N\right) \frac{(rT)^i}{i!}} \\ &= -\frac{2r}{1 - e^{-rT}} \frac{1}{1 - e^{-NrT}} \frac{N^2 \sum_{i=2}^{\infty} \left(N^{2(i-1)} - 1\right) \frac{(rT)^{2i}}{(2i)!}}{N \sum_{i=2}^{\infty} \left(N^{i-1} - 1\right) \frac{(rT)^i}{i!}} \\ &= -\frac{2r}{1 - e^{-rT}} \frac{N}{1 - e^{-NrT}} \frac{\sum_{i=2}^{\infty} \left(N^{2(i-1)} - 1\right) \frac{(rT)^{2i}}{(2i)!}}{\sum_{i=2}^{\infty} \left(N^{i-1} - 1\right) \frac{(rT)^i}{i!}}.\end{aligned}$$

The first factor of this product does not depend on N. The second factor and the third factor are increasing in N if $N \geqslant 2$. Hence, $\Upsilon_N(T)$ decreases in N - note the minus sign - which proves (b).[15]

(c) follows directly from the fact that Υ_N is monotone increasing in N, see (b), and the evaluation of the formulas for Υ_2 and $\lim_{n\to\infty}\Upsilon_n(T)$. The latter is equivalent to $\Upsilon(T) = -\frac{r}{1-e^{-rT}}$, cf. (4.14). That $\Upsilon_2 = -\frac{r}{1+e^{-rT}}$ follows by lengthy but basic algebra.

(d) The upper and the lower bounds of $\Upsilon_N(T)$ converge to $-r$ when T goes to infinity. Thus, $\lim_{T\to\infty}\Upsilon_N(T) = -r$.

(e) To show that $\lim_{r\to 0}\Upsilon_N(T;r) = 0$, we use the representation of the derivatives of S_N as geometric sums, cf. (4.7), (4.8), and (4.9). Since the fractions

$$\frac{\ddot{S}_N(T;r)}{\dot{S}_N(T;r)} = \frac{r^2\sum_{i=1}^{N-1} i^2 e^{-irT}}{-r\sum_{i=1}^{N-1} i\, e^{-irT}} = -r\frac{\sum_{i=1}^{N-1} i^2 e^{-irT}}{\sum_{i=1}^{N-1} i\, e^{-irT}}$$

and

$$\frac{\dot{S}_N(T;r)}{S_N(T;r)} = \frac{-r\sum_{i=1}^{N-1} i\, e^{-irT}}{\sum_{i=0}^{N-1} e^{-irT}}$$

both converge to zero if $r \to 0$, the difference $\Upsilon_N(T;r) = \frac{\ddot{S}_N(T;r)}{\dot{S}_N(T;r)} - \frac{\dot{S}_N(T;r)}{S_N(T;r)}$ converges to zero if $r \to 0$. ♦

Figure 4.1 depicts $S_N(T)$ as functions of T (panel (a)) and the function $\Upsilon_N(T)$ (panel (b)) for different values of N; $r = 0.1$. The functions $S_N(T)$ strictly decrease in N as well as in T. The behavior of the functions $\Upsilon_N(T)$ is more complex. For example, if $N = 2$, $\Upsilon_2(T)$ strictly decreases in T. The function $\Upsilon(T)$, however, strictly increases in T. In general, $\Upsilon_N(T)$, N *sufficiently* large, decreases for *small* values of T, and is increasing if T is *large*. We will use the function Υ_N to formulate conditions for the uniqueness of the optimal cycle length, see below. In particular, we will impose the condition that the logarithmic derivatives of the profit margins are bounded from above by $\Upsilon_N(T)$, respectively $\Upsilon(T)$.

We consider the following decision problem: choose the cycle length $\tau \geqslant 0$ such that the present value of the N-period profit π_N, $N \geqslant 1$,

$$\pi_N(\tau) = (\pi_1^*(\tau) - k)\sum_{i=0}^{N-1} e^{-ir\tau} = (\pi_1^*(\tau) - k)\, S_N(\tau), \tag{4.15}$$

[15] Note, that $\Upsilon_N(T)$ decreases for all real values $N > 1$ and not only integer values as assumed in Proposition 4.2.1.

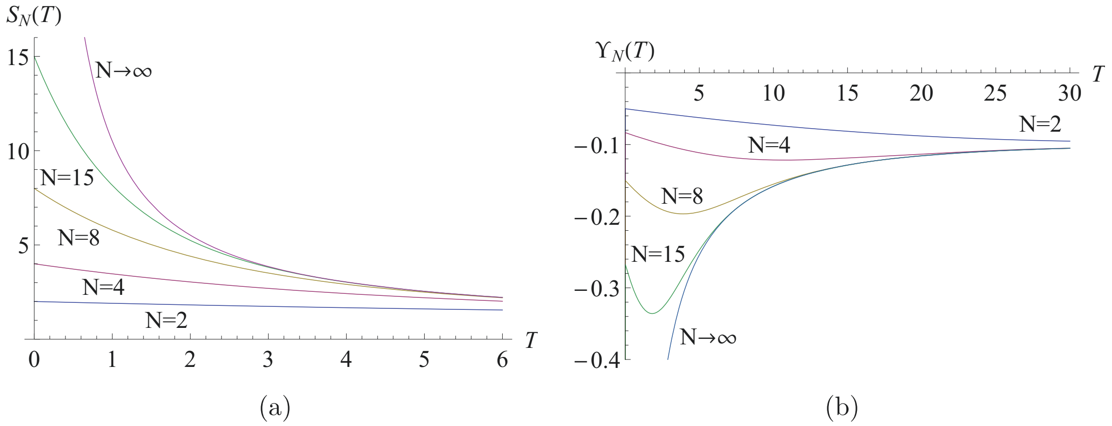


Figure 4.1: The functions $S_N(T)$ (panel (a)) and $\Upsilon_N(T)$ (panel (b)) for different values of N; $r \equiv 0.1$.

is maximized. The present value of the N-period profit is the product of the payoff per period, $\pi_1^*(\tau) - k$, and $S_N(\tau)$; for any τ, this second factor $S_N(\tau)$ is larger than one if $N > 1$, cf. part (i) of Proposition 4.2.1. If $N \equiv 1$, the objective is to simply maximize (with respect to τ) the one-cycle profit. Since the profit rate ν^* is strictly positive, this problem is unbounded: $\pi_1^*(\tau)$ strictly increases in τ and $S_1(\tau) \equiv 1$, see below. We set $\pi_N(0) := 0$. If T_0 - the minimum cycle length according to (4.6) - is infinite, it is not possible to find a positive value τ such that the total profit will be positive. We therefore interpret the value $\tau = 0$ as *no market-entry*, a decision which incurs neither gains nor losses. Note, we do not assume the cycle length to be bounded. The decision maker has to choose a sequence of (equidistant) time intervals[16], but there is no overall time limit in which these events have to occur.

We also consider the infinite cycle problem which is to maximize

$$\pi_\infty(\tau) = (\pi_1^*(\tau) - k) \sum_{i=0}^{\infty} e^{-ir\tau} = (\pi_1^*(\tau) - k)\, S(\tau) \tag{4.16}$$

with respect to $\tau \geqslant 0$; we set $\pi_\infty(0) := 0$. Both functions, $\pi_N(\tau)$ and $\pi_\infty(\tau)$, are the product of two terms: one increases in τ and the other one decreases in τ. Thus, the monopolist has to find a trade-off between increasing the one-cycle profit by choosing τ large and waiting too long. When only looking at the factor $\pi_1^*(\tau) - k$, it is profitable to choose a large value of τ.[17] On the other hand, waiting is penalized by the factor $S_N(\tau)$,

[16]For example, the number of years that pass until a new car model is launched or the interval between the next N entertainment shows of an artist in a city.

[17]The profit of one cycle, $\pi_1^*(\tau)$, strictly increases in τ since $d\pi_1^*(\tau)/d\tau = \nu^*(\tau) > 0$, cf. Theorem 2.2.1

respectively $S(\tau)$; the values of $S_N(\tau)$ and $S(\tau)$ decrease monotonically in τ towards one, cf. Proposition 4.2.1. For $N \geqslant 1$ the largest value that can be attained by the function $S_N(\tau)$ equals N; recall, $\lim_{\tau\to 0} S_N(\tau) = N$. The time value of the periodical payments and the profit per period, the *annuity*, are nonlinear expressions of τ, and the product is also a nonlinear function. Thus, (4.15) and (4.16) are nonlinear objective functions.

Theorem 4.2.1 *Let Condition 4.1.1 and the assumptions of Proposition 4.2.1 hold. Let* ν^* *be defined as in Theorem 2.2.1. Let* $N > 1$, *and assume:*

(i) The first hitting time T_0, *see* (4.6), *exists and is finite.*

(ii) $\lim_{t\to\infty} e^{rt}\nu^*(t) = 0$.

Then,
(a) there exists an optimal cycle length τ_N^*, $T_0 < \tau_N^* < \infty$, *that maximizes* (4.15).

Moreover, if the assumptions (*i*), (*ii*), *and*

(iii) $\mu(t)$ *is a differentiable function on the open interval* (T_0, ∞) *and, for all* $\tau > T_0$,

$$\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} = -\left[r + \frac{1}{1-\Delta}\left((\varepsilon-1)\frac{\dot{c}(\tau)}{c(\tau)} - \frac{\dot{\mu}(\tau)}{\mu(\tau)}\right)\right] < \Upsilon_N(\tau) \tag{4.17}$$

are satisfied, then,
(b) the optimal cycle length τ_N^* *is unique;* τ_N^* *is the (unique) solution of the equation*

$$\pi_1^*(\tau) - k = -\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau). \tag{4.18}$$

Proof. The basic idea of the proof is to show that the function $\pi_N(\tau)$ is continuous on the semi-open interval $[T_0, +\infty)$, it is positive on $(T_0, +\infty)$, and it is decreasing for large values of τ. Consequently, there exists a finite value $\tau_N^* > T_0$ that maximizes $\pi_N(\tau)$.

(a) The function $\pi_1^*(\tau)$ is the integral of the piecewise continuous function ν^*, cf. Theorem 2.2.1. Hence, $\pi_1^*(\tau)$ is continuous, and except at a possibly finite number of points, it is also differentiable. The function $S_N(\tau)$ is continuous and differentiable in τ, cf. Proposition 4.2.1. Thus, the function $\pi_N(\tau)$ is also continuous and differentiable in τ. Since $S_N(\tau) > 1$ for all $\tau > 0$, assumption (i) ensures that the net profit of any cycle of length $\tau > T_0$ is positive. Hence, $\pi_N(\tau)$ is positive on the open interval (T_0, ∞).

in Section 2.2.

At all points where $\pi_N(\tau)$ is differentiable[18], the first derivative of $\pi_N(\tau)$ is given by, τ large,

$$\begin{aligned}\frac{d\pi_N}{d\tau}(\tau) &= \dot{S}_N(\tau)\left(\pi_1^*(\tau)-k\right)+S_N(\tau)\nu^*(\tau)\\ &= \dot{S}_N(\tau)\left[\pi_1^*(\tau)-k+\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau)\right]. \qquad (4.19)\end{aligned}$$

The first factor of the product (4.19), $\dot{S}_N(\tau)$, is strictly smaller than zero, cf. Proposition 4.2.1. The term $\pi_1^*(\tau)-k$ is positive due to assumption (i) and strictly increases in τ since $\nu^*(t)$, the integrand of $\pi_1^*(\tau)$, is positive. Since the expression for $\nu^*(t)$ contains the factor e^{-rt}, cf. equation (2.18), assumption (ii) implies that also $\nu^*(t)$ converges to zero. Thus, $\lim_{\tau\to\infty}\left(e^{r\tau}-1\right)\nu^*(\tau)=0$. Hence,

$$\begin{aligned}\lim_{\tau\to\infty}\left(\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau)\right) &= \lim_{\tau\to\infty}\left(-\frac{e^{r\tau}-1}{r\left(1-N\frac{e^{r\tau}-1}{e^{Nr\tau}-1}\right)}\nu^*(\tau)\right)\\ &= -\frac{1}{r}\lim_{\tau\to\infty}\left(\frac{1}{1-N\frac{e^{r\tau}-1}{e^{Nr\tau}-1}}\right)\lim_{\tau\to\infty}\left(\left(e^{r\tau}-1\right)\nu^*(\tau)\right)\\ &= -\frac{1}{r}\cdot 1\cdot\lim_{\tau\to\infty}\Big(\left(e^{r\tau}-1\right)\nu^*(\tau)\Big)\\ &= 0,\end{aligned}$$

and the second factor in (4.19) is strictly positive for large values τ. Therefore, the first derivative of $\pi_N(\tau)$ is negative for large values τ, and the function $\pi_N(\tau)$ is decreasing if τ is large. Together, these facts imply (a).

In order to prove claim (b), note, that $\nu^*(t)$ is differentiable on (T_0,∞) since, by assumption, $\mu(t)$ is differentiable on (T_0,∞). The first order condition $\frac{\pi_N}{d\tau}(\tau)=0$ can be rewritten as equation (4.18). Since a maximum exists on an open interval there is at least one solution to equation (4.18). The left-hand side of equation (4.18) strictly increases in τ and takes the value zero at T_0. The right-hand side of equation (4.18) takes a positive value at T_0 - both, the term $-S_N(\tau)/\dot{S}_N(\tau)$ and the function $\nu^*(\tau)$, are positive, cf. Proposition 4.2.1 and Theorem 2.2.1 - and strictly decreases on (T_0,∞). To

[18] We want to show that $\pi_N(\tau)$ strictly decreases for large τ values. Since, by assumption, $\pi_N(\tau)$ is continuous, we can neglect the finitely many points where $\pi_N(\tau)$ is not differentiable.

see the latter fact, note, that the derivative[19] of the right-hand side of equation (4.18),

$$\begin{aligned}
\frac{d\left(-\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau)\right)}{d\tau} &= -\left[\frac{\dot{S}_N(\tau)^2 - S_N(\tau)\ddot{S}_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau) + \frac{S_N(\tau)}{\dot{S}_N(\tau)}\dot{\nu}^*(\tau)\right] \\
&= -\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau)\left[\frac{\dot{S}_N(\tau)}{S_N(\tau)} - \frac{\ddot{S}_N(\tau)}{\dot{S}_N(\tau)} + \frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)}\right] \\
&= -\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau)\left[\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} - \Upsilon_N(\tau)\right]
\end{aligned}$$

is negative on (T_0, ∞) due to condition (4.17). Thus, the right-hand side of (4.18) decreases on (T_0, ∞), recall $\dot{S}_N(\tau) < 0$. The solution of (4.18) is unique and so is the maximum of $\pi_N(\tau)$. ♦

Remark 4.2.1 *Although we refer to the optimal profit rate ν^*, cf. Chapter 2, any (positive) profit rate satisfying the assumptions in Theorem 4.2.1 will ensure the existence and uniqueness of an optimal cycle length. In particular, the results of Theorem 4.2.1 also cover the partially static cases where the profit rate is given by $\bar{\nu}$ and $\tilde{\nu}$, respectively. Theorem 4.2.1 also applies to the case when the profit rate is externally given and is not subject to the decision of the inventory manager.*

The *market-entry* condition (i) in Theorem 4.2.1 ensures that a monopolist is able to choose a cycle length with no losses. Technically speaking, assumption (i) implies a lower bound on the (optimal) cycle length. Assumption (ii) ensures that the optimal cycle length is finite. It is not profitable for the monopolist to wait too long before starting a new cycle. Hence, assumption (ii) can be seen as a *cycle-exit* condition. In our case, the optimal profit rate ν^* is given in terms of the present value, cf. equations (2.10) and (2.18). If elasticities are constant, the profit rate (in simplified terms) is given by

$$\nu^*(t) = const \cdot e^{-rt}\left(\frac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \overset{\delta \geq 0}{=} const \cdot e^{-rt} w^*(t)^a. \tag{4.20}$$

Hence, condition (ii) implies that $w^*(t)^a$, or, equivalently, the expression $\left(\frac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}}$, converges to zero if $t \to +\infty$, i.e., the market conditions are not in favor of advertising efforts in the long run.[20] To be precise, assumption (ii) is a restriction on the ratio of the arrival intensity $\mu(t)$ and the cost function $c(t)$ (to the power of $\varepsilon - 1$). Since $c(t)$ is

[19] Since, by assumption, $\mu(t)$ is differentiable on (T_0, ∞), $\nu^*(t)$ is differentiable on (T_0, ∞).
[20] Note, that in condition (ii) the profit rate ν^* is multiplied by the factor e^{-rt}.

monotonically increasing, cf. equations (2.8) and (2.9), if μ is constant, assumption (*ii*) is satisfied.

The condition for the uniqueness of the optimal cycle length, cf. (*iii*), imposes even stronger restrictions on the parameter functions. Asking the log-derivative of the (optimal) profit rate at time τ to be smaller than the value $\Upsilon_N(\tau)$ for all $\tau > T_0$, requires, since $\Upsilon_N(\tau) < 0$, a decreasing profit rate on the open interval $(T_0, +\infty)$.

Theorem 4.2.2 *Let Condition 4.1.1 and all assumptions of Proposition 4.2.1 hold. Let ν^* be defined as in Theorem 2.2.1. Let $N := +\infty$, and assume:*

(i) The first hitting time T_0, see (4.6)*, exists and is finite.*

(ii) $\lim_{t\to\infty} e^{rt}\nu^*(t) = 0$.

Then,

(a) there exists an optimal cycle length τ^*, $T_0 < \tau^* < \infty$, that maximizes (4.16).

Moreover, if the assumptions (*i*), (*ii*), *and*

(iii) $\mu(t)$ is a differentiable function on the open interval (T_0, ∞) and, for all $\tau > T_0$,

$$\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} = -\left[r + \frac{1}{1-\Delta}\left((\varepsilon - 1)\frac{\dot{c}(\tau)}{c(\tau)} - \frac{\dot{\mu}(\tau)}{\mu(\tau)}\right)\right] < -r \tag{4.21}$$

are satisfied, then,

(b) the optimal cycle length τ^* is unique; τ^* is the (unique) solution of the equation

$$\pi_1^*(\tau) - k = \frac{1}{r}\left(e^{r\tau} - 1\right)\nu^*(\tau). \tag{4.22}$$

Proof. The proof follows along the lines of the proof of Theorem 4.2.1. We show that the function $\pi_\infty(\tau)$ is continuous on the semi-open interval $[T_0, +\infty)$, it is positive on $(T_0, +\infty)$, and it is decreasing for large values of τ. Consequently, there exists a finite value $\tau^* > T_0$ that maximizes (4.16).

(*a*) Since $S(\tau)$ is differentiable, the function $\pi_\infty(\tau)$ is continuous and except at a possibly finite number of points also differentiable, cf. the proof of Theorem 4.2.1. The first derivative of $\pi_\infty(\tau)$ is given by

$$\begin{aligned}\dot{\pi}_\infty(\tau) = \frac{\pi_\infty}{d\tau}(\tau) &= \dot{S}(\tau)\left(\pi_1^*(\tau) - k\right) + S(\tau)\nu^*(\tau)\\ &= \dot{S}(\tau)\left[\pi_1^*(\tau) - k + \frac{S(\tau)}{\dot{S}(\tau)}\nu^*(\tau)\right];\end{aligned}$$

the term $\frac{S(\tau)}{\dot{S}(\tau)}$ equals $\frac{e^{r\tau}-1}{r}$, cf. Proposition 4.2.1. The term $\frac{S(\tau)}{\dot{S}(\tau)}\nu^*(\tau)$ converges to zero if τ tends to infinity due to condition (ii). Since $\pi_1^*(\tau) - k > 0$ (assumption (i)) and $\dot{S}(\tau) < 0$, the first derivative of $\pi_\infty(\tau)$ is negative if τ gets large. Hence, the objective function $\pi_\infty(\tau)$ decreases if τ becomes large, and attains its maximum at a finite value $\tau^* > T_0$.

In order to prove (b), first notice that equation (4.22) is equivalent to the first order condition $d\pi(\tau)/d\tau = 0$. Since a maximum exists on an open interval there is at least one solution to equation (4.22). We will show that, assuming (iii), *every* stationary point τ^* satisfies the second order condition of a (local) maximum: $\frac{d^2\pi_\infty}{d\tau^2}(\tau^*) < 0$. Since $\pi_\infty(\tau)$ is continuous this observation implies that τ^* has to be unique as otherwise there must exist a local minimum between any two local maxima.

According to (iii), we assume that $\mu(t)$ is differentiable on $[T_0, +\infty)$, and the derivative $\dot{\nu}^*$ exists. The second derivative of π_∞ evaluated at $\tau^* > T_0$ is given by, cf. Proposition 4.2.1,

$$
\begin{aligned}
\frac{d^2\pi_\infty}{d\tau^2}(\tau^*) &= \ddot{S}(\tau^*)\left(\pi_1^*(\tau^*) - k\right) + 2\dot{S}(\tau^*) + S(\tau^*)\dot{\nu}^*(\tau^*) \\
&= \ddot{S}(\tau^*)\left(-\frac{S(\tau^*)}{\dot{S}(\tau^*)}\nu^*(\tau^*)\right) + 2\dot{S}(\tau^*) + S(\tau^*)\dot{\nu}^*(\tau^*) \\
&= S(\tau^*)\nu^*(\tau^*)\left(-\frac{\ddot{S}(\tau^*)}{\dot{S}(\tau^*)} + 2\frac{\dot{S}(\tau^*)}{S(\tau^*)} + \frac{\dot{\nu}^*(\tau^*)}{\nu^*(\tau^*)}\right) \\
&= S(\tau^*)\nu^*(\tau^*)\left(r\frac{e^{r\tau^*}+1}{e^{r\tau^*}-1} - r\frac{2}{e^{r\tau^*}-1} + \frac{\dot{\nu}^*(\tau^*)}{\nu^*(\tau^*)}\right) \\
&= S(\tau^*)\nu^*(\tau^*)\left(r + \frac{\dot{\nu}^*(\tau^*)}{\nu^*(\tau^*)}\right),
\end{aligned}
$$

where we make use of the first order condition in the second line. Thus, assuming (4.21), the second derivative at every stationary point is negative. Hence, there exists only one solution τ^* of (4.22). ♦

Whether or not an optimal cycle length exists does not depend on the number of cycles considered, since conditions (i) and (ii) of Theorem 4.2.1 and Theorem 4.2.2 are identical. Assumption (iii) of Theorem 4.2.2, however, is different from assumption (iii) of Theorem 4.2.1: since $-r > -rS(T) = \lim_{N\to\infty}\Upsilon_N(\tau)$, the right-hand sides of both assumptions are not equivalent.

The following example illustrates the foregoing theorems. We consider a situation without holding costs, and where the deterioration rate equals zero. Even in this seemingly trivial constellation - only the constant unit and setup costs have to be considered - verifying assumptions (i) to (iii) turns out to be nontrivial.

Example 4.2.1 *Let $\ell(t) = q(t) \equiv 0$, and $r, c_0 > 0$, i.e., $c(t) = c_0 e^{rt}$. Let a, δ, and ε be given. Recall, $\gamma = \frac{\varepsilon-\Delta}{1-\Delta} > \varepsilon > 1$ and $\Delta = \delta/a$. Then, the profit rate ν^* according to* (2.18) *is given by*

$$\nu^*(t) = c_{\nu^*} e^{-\gamma r t} \mu(t)^{\frac{1}{1-\Delta}}, \tag{4.23}$$

where $c_{\nu^*} \equiv \frac{1-\Delta}{\Delta} \left[\frac{\Delta}{\varepsilon-1} \left(\frac{\varepsilon-1}{\varepsilon} \right)^{\varepsilon} \right]^{\frac{1}{1-\Delta}} c_0^{-(\gamma-1)}$.

If $\mu(t)$ is equal to a constant μ which satisfies

$$\frac{\gamma r}{c_{\nu^*}} k \mu^{-\frac{1}{1-\Delta}} < 1, \tag{4.24}$$

then, a finite value of the minimum cycle length T_0 exists. This value is given by

$$T_0 = -\frac{1}{\gamma r} \log \left(1 - \frac{\gamma r}{c_{\nu^*}} k \mu^{-\frac{1}{1-\Delta}} \right). \tag{4.25}$$

Moreover, if μ satisfies (4.24)*, there exist optimal cycle length values τ_N^* and τ^*.*

In general, to check assumption (i) - the market-entry condition - it suffices to identify at least one value τ such that $\pi_1(\tau) > k$. To do so, one can numerically evaluate the integral $\pi_1(\tau) = \int_0^{\tau} \nu^*(t)dt$ for large values of τ. In Example 4.2.1, if the arrival rate μ is constant and satisfies (4.24), we find the closed form expression (4.25) of the value T_0 depending on the parameter values of the model.[21] In this special case, (4.25) can also be used to analyze the dependence of the minimum cycle length on the model parameters.

Condition (ii) of both Theorems 4.2.1 and 4.2.2 - the cycle-exit condition - requires ν^*, see (4.23), to satisfy

$$\lim_{t\to\infty} e^{rt} \nu^*(t) = c_{\nu^*} \lim_{t\to\infty} e^{-(\gamma-1)rt} \mu(t)^{\frac{1}{1-\Delta}} = 0. \tag{4.26}$$

Since $\gamma > 1$, (4.26) holds true whenever μ is bounded from above, or if μ is any polynomial function. Moreover, μ is allowed to increase or fluctuate as long as $\mu(t)^{1/(1-\Delta)}$ is dominated by the term $e^{-(\gamma-1)rt}$ when $t \to +\infty$. In particular, (4.26) holds true if the arrival rate μ is constant and $r > 0$. Since both conditions, the market-entry and the cycle-exit condition, are satisfied, there exist optimal cycle length values τ_N^* and τ^*.

[21] Elementary calculations yield the equation (4.25) as the solution of $\int_0^{T_0} \nu^*(t)dt = k$, where $\int_0^{T_0} \nu^*(t)dt = c_{\nu^*} \mu^{\frac{1}{1-\Delta}} \int_0^{T_0} e^{-\gamma r t} dt = c_{\nu^*} \mu^{\frac{1}{1-\Delta}} \frac{1}{\gamma r} \left(1 - e^{-\gamma r T_0}\right)$. Note, for instance, if k is too large so that (4.24) is not satisfied, no finite T_0 exists.

To check assumption (*iii*) of Theorem 4.2.1 and Theorem 4.2.2, we compute the logarithmic derivative of ν^*:

$$\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} = \frac{1}{1-\Delta}\frac{\dot{\mu}(\tau)}{\mu(\tau)} - \gamma r. \tag{4.27}$$

Recall, for all $\tau > T_0$, the logarithmic derivative of ν^* must be smaller than $\Upsilon_N(\tau)$ in the case of a finite cycle problem and smaller than $-r$ if N is infinite. Obviously, if $N = \infty$ and $\mu(t)$ is nonincreasing, then $\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} \leqslant -\gamma r < -r$. Moreover, whenever μ satisfies

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{d \log \mu(t)}{dt} < (\varepsilon - 1)r \tag{4.28}$$

for all $t > T_0$, there is a unique τ^*. For instance, the inequality (4.28) is satisfied if $\mu(t) = \mu_0 \cdot e^{\frac{\varepsilon-1}{2}rt}$, $\mu_0 > 0$. Condition (4.28), however, is violated, if μ is a linear function in time and T_0 is too small, e.g., $\mu(t) = \mu_0 + \mu_1 t$, $\mu_0, \mu_1 > 0$, and $T_0 < \frac{1}{(\varepsilon-1)r} - \frac{\mu_0}{\mu_1}$.

Next, we delve into Example 4.2.1 by considering a numerical example and different scenarios for the μ function.

Example 4.2.1 (continued) *Let* $r = 0.1, c_0 = 1, \ell = q = 0, \varepsilon = 2, a = 2, \delta = 1$, *and* $k = 20$. *These values imply* $\Delta = 0.5$ *and* $\gamma = 3$. *Then, the constant* c_{ν^*}, *see* (4.23), *becomes* $1/64$ *and* $c(t)$ *equals* $e^{0.1t}$. *We consider four different* μ *functions* (I) *to* (IV), *see the second column in Table 4.1. The associated (optimal) profit margins, cf.* (4.23), *are given in the third column.*

scenario	arrival intensity	profit margin
(I)	$\mu(t) = \mu_I \equiv 25$	$\nu^*(t) \approx 9.77e^{-0.3t}$
(II)	$\mu(t) = \mu_{II} = 1.1\mu_I \equiv 27.5$	$\nu^*(t) \approx 11.82e^{-0.3t}$
(III)	$\mu(t) = \mu_I\,(1 + 0.5\sin(t))$	$\nu^*(t) \approx 9.77(1 + 0.5\sin(t))^2 e^{-0.3t}$
(IV)	$\mu(t) = \mu_I \frac{e^{(\gamma-1.5)rt}}{1+(\gamma-2)rt} = 25\frac{e^{0.15t}}{1+0.1t}$	$\nu^*(t) \approx \frac{9.77}{(1+0.1t)^2}$

Table 4.1: The μ function for each scenario (2nd column) and the associated (approximate) profit margin according to (4.23). Other parameter values are $r = 0.1, c_0 = 1, \ell = q = 0$, $\varepsilon = 2, a = 2, \delta = 1$, and $k = 20$.

In scenarios (I) and (II), the arrival rates are constant ($\mu_{II} > \mu_I$). The cases (III) and (IV) illustrate a time-inhomogeneous customer behavior. Scenario (III) represents a stylized (perfect) seasonal pattern; the demand oscillates (with frequency $\frac{1}{2\pi}$) between the values 12.5 and 37.5. In scenario (IV), more and more customers arrive over time. The arrival rate is an exponential function damped by a linear time dependent expression.

Panel (a) of Figure 4.2 depicts the different scenarios on the time interval $[0, 15]$. In the following, we will investigate each scenario and concentrate on the existence of an optimal cycle length in the special cases when the number of cycles is three or infinity, i.e., $N = 3$ or $N = +\infty$.

The cost function $c(t)$ is of the simple exponential type $c(t) = e^{0.1t}$. If $\mu(t)$ is also a (rather) simple expression, for example a constant, one is able to explicitly compute the integral of $\nu^*(t)$ - the one-cycle profit (depending on $\tau > 0$). In scenario (I), for instance, $\pi_1^*(\tau) = 3125/96\left(1 - e^{-3\tau/10}\right) - 20$, and the root of this expression, the minimum cycle length T_0, can be easily computed, cf. (4.25). The second column of Table 4.2 gives the corresponding T_0 values (with a two digit precision) for each scenario. Assumption (i) is thus satisfied for all four scenarios: in any of the particular market environments it is profitable to enter the market.

The cycle-exit condition (ii) requires the expression $e^{rt}\nu^*(t)$ to converge to zero. This condition is satisfied in scenarios (I), (II), and (III). In scenario (IV), no finite optimal cycle length τ_N^*, N finite or infinite, exists. The reason is that $e^{rt}\nu^*(t) \approx 9.77\frac{e^{0.1t}}{(1+0.1t)^2}$, and this function tends to infinity if $t \to \infty$. Panel (b) of Figure 4.2 shows the graphs of the total profit functions $\pi_N(\tau)$ for all four scenarios if $N = 3$; panel (c) of Figure 4.2 shows the graphs of the total profit functions if $N = \infty$. In both windows, the plots of both profit functions for scenario (IV) indicate that the profit functions strictly increase in τ, i.e., it is profitable to stay in the market forever. Moreover, the graphs of panel (b) and (c) suggest that only in scenarios (I) and (II) the optimal cycle length is unique. One easily verifies that condition (4.18) is indeed satisfied for all $\tau > 0$ if the arrival intensity is constant:

$$\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} \overset{(I,II)}{=} -\gamma r = -3r = -3r\frac{2 + 3e^{r\tau} + 3e^{2r\tau} + e^{3r\tau}}{2 + 3e^{r\tau} + 3e^{2r\tau} + e^{3r\tau}} = -r\frac{6 + 9e^{r\tau} + 9e^{2r\tau} + 3e^{3r\tau}}{2 + 3e^{r\tau} + 3e^{2r\tau} + e^{3r\tau}}$$

$$< -r\frac{e^{r\tau} + 4e^{2r\tau} + e^{3r\tau}}{2 + 3e^{r\tau} + 3e^{2r\tau} + e^{3r\tau}} = -r\frac{2e^{2r\tau}\left(2 + \cosh(r\tau)\right)}{(2 + e^{r\tau})(1 + e^{r\tau} + e^{2r\tau})} = \Upsilon_3(\tau).$$

Since $\gamma > 1$, also inequality (4.21), $\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} < -r$, is satisfied. Thus, τ^* is unique.[22] In scenario (III), the condition for uniqueness fails to be satisfied. However, the graph of the profit functions in Figure 4.2 suggests that unique values τ_3^* and τ^* exist. These values can actually be determined numerically. For the scenarios (I) to (IV), Table 4.2

[22] Note, that the uniqueness of τ_N^* and τ^* for scenario (I) and (II) does not depend on the specific values a, δ, and ε, since γ is always bigger than one.

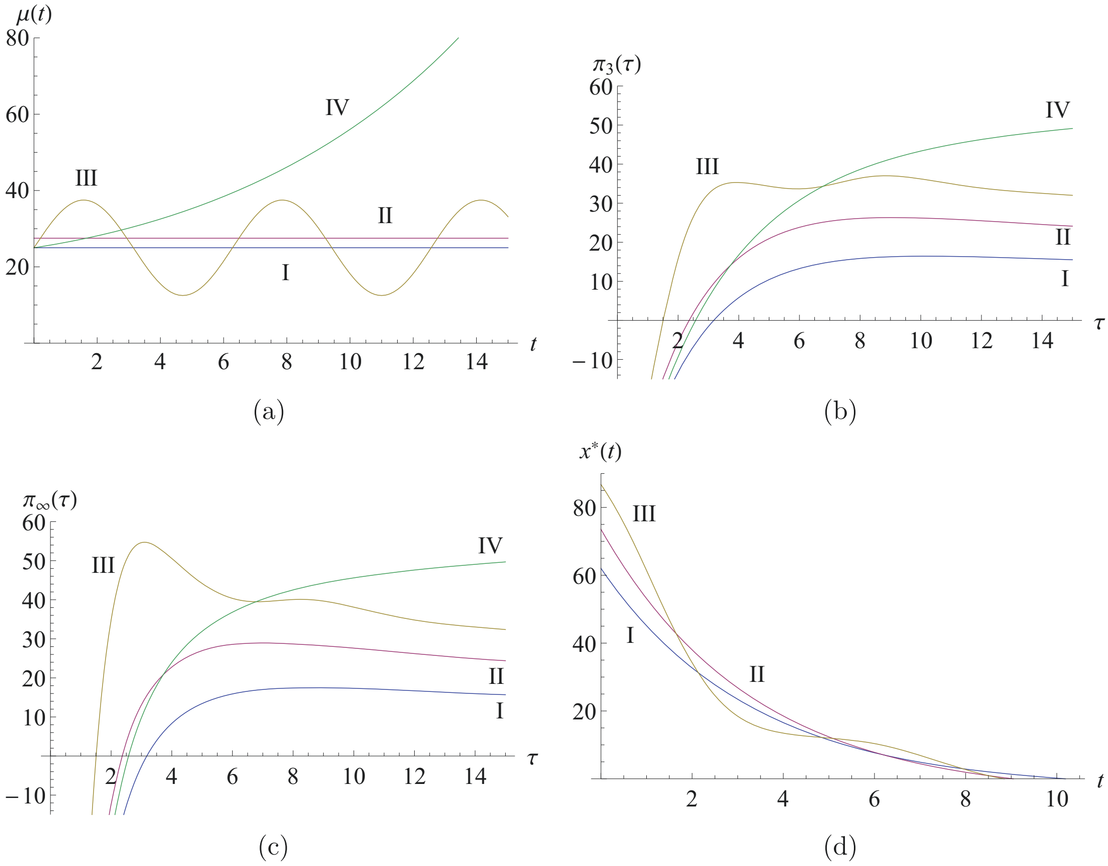


Figure 4.2: The 3-cycle profit $\pi_3(\tau)$, panel (b), and the infinite-cycle profit $\pi_\infty(\tau)$, panel (c), as functions of τ for the different μ scenarios $(I)-(IV)$, panel (a). Panel (d) shows the inventory process of one optimally controlled cycle for $N = 3$ for scenarios $(I)-(III)$, cf. Table 4.2. All plots assume $r = 0.1, c_0 = 1, \ell = q = 0, \varepsilon = 2, a = 2, \delta = 1$, and $k = 20$.

displays the optimal cycle length, the associated (optimal) capacity, cf. (2.7),

$$x_N^* = \int_0^{\tau_N^*} \lambda^*(t)dt,$$

and the optimal profit values (for $N = 3$ and $N = +\infty$). The numbers in Table 4.2 and the plots of Figure 4.2 suggest that the optimal cycle length becomes smaller if μ gets larger, cf. scenario (I) and (II), but also scenario (III), where the arrival rate lies above its *average* value μ_I at the beginning.[23] The fourth and the seventh column of Table 4.2 show the capacity (or initial inventory) associated with the optimal control and optimal

[23]Recall, in scenario (III), the function $\mu(t)$ oscillates around its mean value $\mu_I = 25$.

		$N=3$			$N\to\infty$		
scenario	T_0	τ_3^*	x_3^*	$\pi_3(\tau_3^*)$	τ^*	x_∞^*	$\pi_\infty(\tau_\infty)$
(I)	3.18	10.19	62.04	16.44	8.60	60.17	17.48
(II)	2.36	9.05	73.55	26.31	7.04	69.23	28.93
(III)	1.51	8.85	86.76	37.02	3.10	69.17	54.71
(IV)	2.83	$+\infty$	195.31	77.66	$+\infty$	195.31	77.66

Table 4.2: Minimum cycle lengths, optimal cycle lengths, and capacities and total profits associated with the optimal cycle lengths for the scenarios $(I)-(IV)$; parameter values are $r=0.1, c_0=1, \ell=q=0, \varepsilon=2, a=2, \delta=1$, and $k=20$.

cycle length. For scenarios (I) to (III) and $N=3$, the optimal cycle length decreases while the associated capacity increases. Expressed in everyday language, in those cases more goods are sold within a shorter time span. This effect can also be observed when $N=+\infty$. If $N=3$, panel (d) of Figure 4.2 shows the optimally controlled inventory processes over one cycle.

Finally, we take a second look at scenario (IV). In the case of scenario (IV), condition (ii) of Theorem 4.2.1 is not satisfied, thus, it is profitable to extend the sales horizon until infinity ($\tau_N^*=+\infty$). In this case, the benefit from waiting is larger than the loss of waiting. The profit rate $\nu^*(t)\approx\frac{9.77}{(1+0.1t)^2}$ decreases over time and converges to zero at a quadratic rate. Hence, the value of the integral $\int_0^\infty \nu^*(t)dt$ is bounded from above, and a limit for the net present value of the (optimal) total profit exists. Table 4.2 gives the corresponding values of $\pi_3(\tau_3^*)$ and $\pi_\infty(\tau^*)$ for the cases $N=3$ and $N=\infty$.

Example 4.2.1 indicates that even in a seemingly simple setting - only unit costs are considered and neither inventory costs nor depreciation effects are taken into account - checking the assumptions of Theorem 4.2.1 and Theorem 4.2.2 can be tricky, cf. scenario (III). For any particular parameter setting, these assumptions have to be analyzed individually. However, we characterize classes of settings that may help to facilitate this analysis. The first part of the following Corollary presents an explicit solution formula for the minimum cycle length T_0 for a specific class of parameter settings (part (a)), cf. (4.25). Furthermore, for special cases ($N=2$ and $N=\infty$) assumption (iii) of Theorem 4.2.1, respectively Theorem 4.2.2, simplifies, see part (b). The second part of Corollary 4.2.1 is a list of structural properties of the optimal cycle length and of the optimal capacity as functions of the number of cycles N and the fixed order cost k.

Corollary 4.2.1 *Let Assumption 4.1.1 and Proposition 4.2.1 hold true, and let ν^* be given by Theorem 2.2.1.*

1. a) *Let $q(t) = \ell(t) = 0$, $\mu(t) \equiv \mu$ and $\nu^*(t) = c_{T_0}\left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} e^{-\gamma r t}$, where $c_{T_0} = \frac{c_\lambda}{\gamma-1} = \frac{1-\Delta}{\Delta}\left[\frac{\Delta}{\varepsilon-1}\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon}\right]^{\frac{1}{1-\Delta}}$. If $k < \frac{c_{T_0}}{\gamma r}\left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}}$, then a finite minimum cycle length T_0 exists:*

$$T_0 = -\frac{1}{\gamma r}\log\left(1 - \frac{\gamma r}{c_{T_0}}\left(\frac{c_0^{\varepsilon-1}}{\mu}\right)^{\frac{1}{1-\Delta}} k\right). \tag{4.29}$$

   b) *Let $q(t), \ell(t) \geqslant 0$ be continuous functions and let T_0 be finite. Let $\rho < (\varepsilon-1)r$ be arbitrary but fixed. If*

$$\frac{\dot{\mu}(t)}{\mu(t)} < \rho \quad \textit{for all } t > T_0, \tag{4.30}$$

   and if $N = 2$ or $N = \infty$, then a unique cycle length exists that maximizes $\pi_N(\tau)$.

2. *Let all assumptions of Theorem 4.2.1 and Theorem 4.2.2 hold true. Then,*
   a) *the optimal cycle length decreases in $N > 1$, i.e., $\tau_N^* > \tau_{N+1}^* > \tau_\infty$,*
   b) *the optimal capacity $x_N^* := \int_0^{\tau_N^*} \lambda^*(t)dt$ decreases in N,*
   c) *the optimal cycle length and the optimal capacity both decrease in k.*

Proof. 1.(a) If $\nu^*(t) = c_{T_0}\left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} e^{-\gamma r t}$, then the profit of one cycle is given by

$$\pi_1^*(\tau) = \int_0^\tau \nu^*(t)dt = c_{T_0}\left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \frac{1}{\gamma r}\left(1 - e^{-\gamma r \tau}\right).$$

If $k < \frac{c_{T_0}}{\gamma r}\left(\frac{\mu}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}}$, then the equation $\pi_1^*(\tau) = k$ has a unique solution since the term $(1 - e^{-\gamma r T})$ monotonically increases in $T \geqslant 0$. Elementary calculations yield the expression (4.29) of the solution value.

(b) Since a finite T_0 exists, the market entry condition (i), see Theorem 4.2.1 and the subsequent discussion, is satisfied. The derivative of the cost function $c(t)$ is given by $\dot{c}(t) = (q(t) + r)\, c(t) + \ell(t)$, cf. (2.9). If $q(t), \ell(t) \geqslant 0$ and $r > 0$, then $c(t)$ is positive and $\frac{\dot{c}(t)}{c(t)} = q(t) + r + \frac{\ell(t)}{c(t)} \geqslant r$, $t \geqslant 0$, follows. Since μ satisfies condition (4.30), we get for any

$\tau \geqslant T_0$,

$$
\begin{aligned}
\frac{\dot{\nu}^*(\tau)}{\nu^*(\tau)} &= -\left[r + \frac{1}{1-\Delta}\left((\varepsilon-1)\frac{\dot{c}(\tau)}{c(\tau)} - \frac{\dot{\mu}(\tau)}{\mu(\tau)}\right)\right] \\
&\leqslant -\left[r + \frac{1}{1-\Delta}\left((\varepsilon-1)r - \frac{d\log\mu(\tau)}{d\tau}\right)\right] \\
&< -\left[r + \frac{1}{1-\Delta}\big((\varepsilon-1)r - \rho\big)\right] \\
&< -\left[r + \frac{1}{1-\Delta}\big((\varepsilon-1)r - (\varepsilon-1)r\big)\right] \\
&= -r.
\end{aligned}
$$

Hence, the sufficient condition (4.21) is satisfied. Since $\Upsilon_2(\tau) = -\frac{r}{1+e^{-r\tau}} > -r$, condition (4.17) is also satisfied. We still have to show that based on the given assumptions condition (ii), namely $\lim_{t\to\infty} e^{rt}\nu^*(t) = 0$, is satisfied. Using Gronwall's lemma, condition (4.30) implies that $\mu(t)$ is bounded from above, i.e., $\mu(t) < \mu(0)e^{\rho t}$ for all $t \geqslant T_0$. Since $\frac{\dot{c}(t)}{c(t)} \geqslant r$, $t \geqslant 0$, the cost function is bounded from below, $c(t) \geqslant c(0)e^{rt} = c_0 e^{rt}$. Hence, $t \geqslant T_0$,

$$
\begin{aligned}
e^{rt}\nu^*(t) &= e^{rt} c_{T_0} e^{-rt}\left(\frac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \\
&= c_{T_0}\left(\frac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \\
&\leqslant c_{T_0}\left(\frac{\mu(t)}{(c_0 e^{rt})^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \\
&< c_{T_0}\left(\frac{\mu(0)e^{\rho t}}{(c_0 e^{rt})^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} \\
&= c_{T_0}\left(\frac{\mu(0)}{c_0^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} e^{\frac{(\rho-(\varepsilon-1)r)}{1-\Delta}t}.
\end{aligned}
$$

Since $\rho < (\varepsilon-1)r$, the expression in the last line converges to zero. Thus, all assumptions of Theorem 4.2.1 and Theorem 4.2.2 are satisfied and a unique optimal cycle length exists.

2. Before we prove claims (a), (b), and (c), we collect facts about both sides of equation (4.18). Since the assumptions of Theorem 4.2.1 are satisfied, for each $N > 1$, a unique

and finite value $\tau_N^* > T_0$ exists, i.e., the equation (4.18),

$$\pi_1^*(\tau) - k = -\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau),$$

has a unique solution τ_N^*. In particular, we are going to exploit the facts that the left-hand side of (4.18) is strictly monotone increasing in τ and that the right-hand side of (4.18) is strictly monotone decreasing in $\tau > T_0$ and in $N > 1$ (separately).

The left-hand side of equation (4.18) strictly increases in τ, since $\pi_1^*(\tau)$ is the integral of the positive function ν^*. In the proof of Theorem 4.2.1, we showed that the right-hand side of (4.18) is strictly monotone decreasing in $\tau > T_0$ under assumption (iii). To show that the right-hand side of (4.18) decreases in N we have to show that the term

$$-\frac{S_N(\tau)}{\dot{S}_N(\tau)} = \frac{e^{r\tau} - 1}{r\left(1 - (e^{r\tau} - 1)\frac{N}{e^{Nr\tau}-1}\right)} \tag{4.31}$$

decreases in N; note, ν^* does not depend on the number of cycles. Since the expression $\frac{N}{e^{Nr\tau}-1}$ in the denominator decreases in $N > 1$[24], the denominator increases in N and thus (4.31) is decreasing in N, i.e., $-\frac{S_N(\tau)}{\dot{S}_N(\tau)}\nu^*(\tau) > -\frac{S_{N+1}(\tau)}{\dot{S}_{N+1}(\tau)}\nu^*(\tau)$ for all $N > 1$ follows. With these preparations we are able to prove 2.(a).

(a) We show that $\tau_N^* \leqslant \tau_{N+1}^*$ leads to a contradiction. To this end, assume $\tau_N^* \leqslant \tau_{N+1}^*$. Then, exploiting the properties of (4.18),

$$\begin{aligned}\pi_1^*(\tau_N^*) - k \leqslant \pi_1^*(\tau_{N+1}^*) - k = -\frac{S_{N+1}(\tau_{N+1}^*)}{\dot{S}_{N+1}(\tau_{N+1}^*)}\nu^*(\tau_{N+1}^*) \leqslant -\frac{S_{N+1}(\tau_N^*)}{\dot{S}_{N+1}(\tau_N^*)}\nu^*(\tau_N^*)\\ < -\frac{S_N(\tau_N^*)}{\dot{S}_N(\tau_N^*)}\nu^*(\tau_N^*) = \pi_1^*(\tau_N^*) - k,\end{aligned}$$

which obviously is a contradiction. Hence, $\tau_N^* > \tau_{N+1}^*$. Since this strict inequality is true for all $N > 1$, we get $\tau_N^* > \tau_{N+1}^* > \lim_{N\to\infty}\tau_N^* = \tau_\infty$. Panel (a) of Figure 4.3 illustrates the proof of statement 2.(a).

(b) The (optimal) sales rate $\lambda^* > 0$ does not depend on the cycle length and not on the number of cycles, cf. Theorem (2.2.1). The capacity associated with the optimal cycle length τ_N^* is given by $x_N^* := \int_0^{\tau_N^*}\lambda^*(t)dt$. If $\tau_N^* > \tau_{N+1}^*$, see (a), then

$$x_N^* = \int_0^{\tau_N^*}\lambda^*(t)dt = \int_0^{\tau_{N+1}^*}\lambda^*(t)dt + \int_{\tau_N^*}^{\tau_{N+1}^*}\lambda^*(t)dt = x_{N+1}^* + \int_{\tau_N^*}^{\tau_{N+1}^*}\lambda^*(t)dt > x_{N+1}^*.$$

[24] Note, $\frac{N}{e^{Nr\tau}-1} = \frac{1}{r\tau}\cdot\frac{1}{1+\frac{Nr\tau}{2!}+\frac{(Nr\tau)^2}{3!}+\ldots}$.

(c) Let $0 < k_1 < k_2$ denote two distinct values of setup cost and, ceteris paribus, let τ_{k_1} and τ_{k_2} denote the associated optimal cycle lengths. Assume $\tau_{k_1} \leqslant \tau_{k_2}$. Then,

$$\begin{aligned} \pi_1^*(\tau_{k_2}) - k_2 \;\; &\leqslant \pi_1^*(\tau_{k_1}) - k_2 \;<\; \pi_1^*(\tau_{k_1}) - k_1 \;=\; -\frac{S_N(\tau_{k_1})}{\dot{S}_N(\tau_{k_1})}\nu^*(\tau_{k_1}) \\ &\leqslant -\frac{S_N(\tau_{k_2})}{\dot{S}_N(\tau_{k_2})}\nu^*(\tau_{k_2}) \;=\; \pi_1^*(\tau_{k_2}) - k_2 \end{aligned}$$

is a contradiction! Hence, $\tau_{k_1} > \tau_{k_2}$. Panel (b) of Figure 4.3 illustrates the proof of case (c). ♦

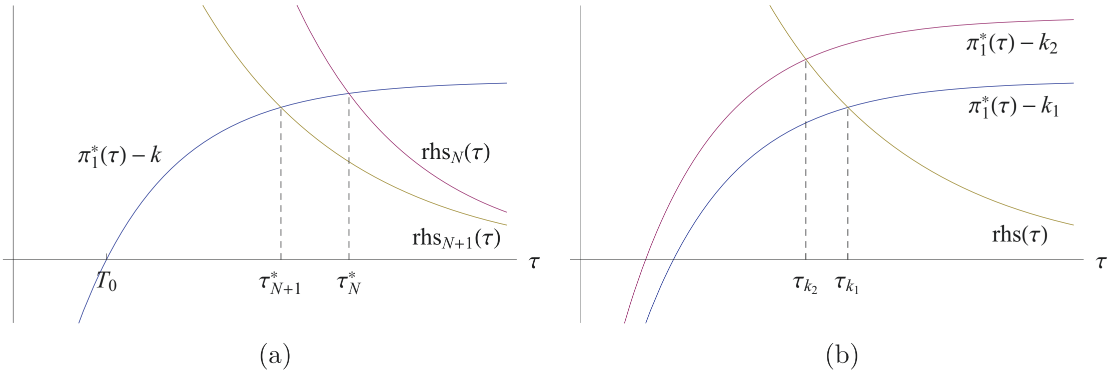


Figure 4.3: Illustration of part 2 of Corollary 4.2.1. Panel (a) shows the left-hand side ($\pi_1^*(\tau) - k$) and the right-hand side of equation (4.18) as functions of τ for N and $N+1$. Panel (b) shows the left-hand side and the right-hand side of equation (4.18) as functions of τ if $k_1 < k_2, k_1 > 0$.

The first part of Corollary 4.2.1 specifies parameter settings that guarantee the existence of a minimum cycle length T_0. In general, it has to be checked (numerically) whether or not $\int_0^\infty \nu^*(t)dt > k$. In part ($b$), conditions are given that guarantee the existence and uniqueness of an optimal cycle length for the two extreme cases $N = 2$ and $N \to +\infty$. For instance, (b) implies that, whenever the arrival rate μ is a nonincreasing function, a unique optimal cycle length exists (assuming $T_0 < +\infty$). The second part of Corollary 4.2.1 summarizes properties of the optimal cycle length and the associated capacity. In the infinite cycle problem, the optimal cycle length and the associated capacity are both smaller than the corresponding values for any finite N. Increased order costs induce shorter optimal selling periods, and thus reduced order capacities. The same holds true if k represents the setup cost in a production problem. In the following section, we analyze the problem of maximizing the average profit.

4.2.2 Maximizing the Average Profit per Time Unit

We now consider the average profit per time unit, the *time-honored* optimization criterion in inventory management. We assume $r(t) \equiv 0$. Many of the following results remain valid assuming a positive interest rate; occasionally, conditions might have to be adjusted to account for $r(t) > 0$. In this subsection, we abstain from explicitly considering the *odd* case of discounted average profits to focus on the influences of the cost parameters on the optimal cycle length.[25]

The profit associated with a sales period of length τ is given by $\pi_1(\tau) = \pi_1^*(\tau) - k = \int_0^\tau \nu^*(t)dt - k$, cf. (4.5) and Theorem 2.2.1 for the formula of ν^*. Recall, ν^* is strictly positive and does not depend on the length of a cycle.[26] The profit per time unit - the (time) average profit - is then given by

$$\pi_\varnothing(\tau) := \frac{\pi_1(\tau)}{\tau} = \frac{1}{\tau}\left(\pi_1^*(\tau) - k\right) = \frac{1}{\tau}\left(\int_0^\tau \nu^*(t)dt - k\right). \tag{4.32}$$

For $\tau \geqslant T_0$, we assume that the average profit is bounded from below and bounded from above. If T_0 is finite, then $\pi_1(T_0) = 0$, see (4.6), and since ν^* is strictly positive, $\pi_1(\tau) > 0$ for all $\tau > T_0$. Thus, zero is a lower bound of the average profit. Let $\limsup_{\tau\to+\infty} \pi_1(\tau)/\tau = 0$, i.e., there is no benefit from *waiting* an infinite time. As in the discounted case, we assume the parameter functions to be identical on every cycle, cf. Condition 4.1.1. The problem to be considered is to choose a cycle length τ such that $\pi_\varnothing(\tau)$ is maximized; we denote such an optimal cycle length by $\tau_\varnothing$. Similar to the discounted case in Section 4.2.1, we will state assumptions that guarantee the existence and uniqueness of an optimal cycle length.

Theorem 4.2.3 *Let Condition 4.1.1 hold, and let ν^* be given by Theorem 2.2.1. Let $r(t) \equiv 0$ for all $t \geqslant 0$, and assume:*

(i) the first hitting time T_0, see (4.6)*, exists and is finite.*

(ii) $\lim_{t\to\infty} \nu^*(t) = 0$.

Then,

(a) there exists an optimal cycle length $\tau_\varnothing$, $T_0 < \tau_\varnothing < +\infty$, that maximizes (4.32).

[25] In Subsection 4.3.2, we allow for discounted average profits. There, we exploit formulas and expressions that hold true for $r(t) \equiv r \geqslant 0$; see, for example, Propositions 4.3.1 and 4.3.2.

[26] The cycle length only determines how long the (optimal) controls are applied within a cycle of length τ.

Moreover, if the assumptions (i)*,* (ii) *and*

(iii) $\mu(t)$ *is a differentiable function on the open interval* $(T_0, +\infty)$*, and*

$$\dot{\nu}^*(t) = -\frac{\nu^*(t)}{1-\Delta}\left[(\varepsilon - 1)\frac{\dot{c}(t)}{c(t)} - \frac{\dot{\mu}(t)}{\mu(t)}\right] \tag{4.33}$$

is negative on $(T_0, +\infty)$*,*

are satisfied, then,

(b) the optimal cycle length $\tau_\varnothing$ *is unique;* $\tau_\varnothing$ *is the (unique) solution of the equation*

$$\pi_\varnothing(\tau) = \nu^*(\tau). \tag{4.34}$$

Proof. The basic idea of the proof is to show that the function $\pi_\varnothing(\tau)$ is continuous on the semi-open interval $[T_0, +\infty)$, it is positive on $(T_0, +\infty)$, and it converges to zero for large values of τ. Consequently, there exists a finite value $\tau_\varnothing > T_0$ that maximizes $\pi_\varnothing(\tau)$.

(a) The (optimal) one-cycle profit $\pi_1^*(\tau)$ is differentiable, cf. the proof of Theorem 4.2.1. Thus, the average profit $\pi_\varnothing(\tau)$ is also a differentiable (and continuous) function. Assumption (i) ensures that $\pi_1^*(\tau) > k$ for all $\tau > T_0$. Hence, $\pi_\varnothing(\tau)$ is positive on the open interval $(T_0, +\infty)$. Assumption (ii) implies that the rate of growth of $\pi_1^*(\tau) = \int_0^\tau \nu^*(t)dt$ tends to zero for large values of τ. Hence, $\pi_\varnothing(\tau) = (\pi_1^*(\tau) - k)/\tau$ converges to zero if τ tends to infinity. Thus, since $\pi_\varnothing$ is continuous, $\pi_\varnothing$ attains its maximum on the interval $(T_0, +\infty)$. This proves statement (a).

(b) By assumption, $\mu(t)$ is a differentiable function on $(T_0, +\infty)$. Hence, $\nu^*(t)$ is a differentiable function on $(T_0, +\infty)$, and the first derivative of $\pi_\varnothing$,

$$\frac{d\pi_\varnothing}{d\tau}(\tau) = \frac{\nu^*(\tau)\tau - (\pi_1^*(\tau) - k)}{\tau^2} = \frac{\nu^*(\tau) - \pi_\varnothing(\tau)}{\tau},$$

is differentiable (and continuous) in τ, too. Since a maximum exists on the open interval $(T_0, +\infty)$, equation (4.34) is equivalent to the first order condition $\frac{d\pi_\varnothing}{d\tau}(\tau) \overset{!}{=} 0$, and the equation is satisfied for (at least) one finite value $\tau_\varnothing > T_0$. Assuming that $\dot{\nu}^*(t)$ is negative on $(T_0, +\infty)$, see (4.33), the second derivative of $\pi_\varnothing$ evaluated at $\tau_\varnothing$,

$$\frac{d^2\pi_\varnothing}{d\tau^2}(\tau_\varnothing) = \frac{\left(\dot{\nu}^*(\tau_\varnothing) - \frac{d\pi_\varnothing}{d\tau}(\tau_\varnothing)\right)\tau_\varnothing - 2\left(\nu^*(\tau_\varnothing) - \pi_\varnothing(\tau_\varnothing)\right)\tau_\varnothing}{\tau_\varnothing^4} = \frac{\dot{\nu}^*(\tau_\varnothing)}{\tau_\varnothing},$$

is negative on $(T_0, +\infty)$. Thus, the critical point $\tau_\varnothing$ is a local maximum. Since *every* critical point satisfies the sufficient condition for being a (local) maximum, there can not be a local minimum point and hence, no two or more local maxima exist. Hence, the point $\tau_\varnothing$ is unique. ♦

Remark 4.2.2 *Although we refer to the optimal profit rate ν^*, cf. Chapter 2, any (positive) profit rate satisfying the assumptions in Theorem 4.2.3 will ensure the existence and uniqueness of the optimal cycle length.*

The assumptions of Theorem 4.2.3 are similar to those of Theorem 4.2.2. The first order condition (4.34) is equivalent to (4.22) when $r \to 0$. Moreover, the conditions imposed on ν^* by (iii) are identical to those in Theorem 4.2.2 if $r \to 0$. One should keep in mind, however, that the factor e^{-rt} which appears in the cost function as well as in the profit rate ν^* becomes one since we assume $r(t) \equiv 0$. If the profit rate is nondecreasing, the existence of an average-optimal cycle length can not be guaranteed. For instance, in the time-homogeneous setting without deterioration and inventory cost, i.e., $\mu(t) \equiv \mu, \ell(t) = q(t) \equiv 0$ for all $t \geqslant 0$, the optimal profit rate $\nu^*(t) \equiv \nu^*$ is a constant and a finite average profit maximizing cycle length $\tau_\varnothing$ does not exist and $\lim_{\tau\to\infty} \pi_\varnothing(\tau) = \nu^*$.[27] Panel (a) of Figure 4.4 illustrates this time-homogeneous situation; for simplicity, we set

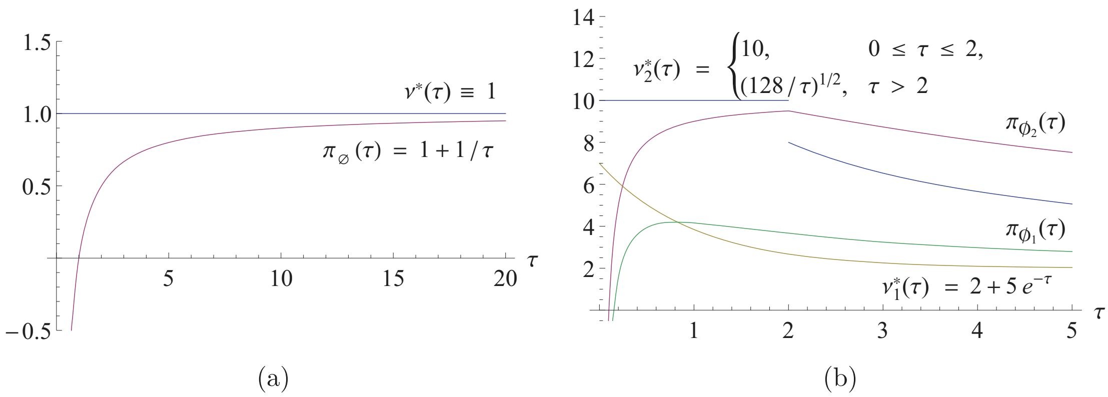


Figure 4.4: Illustration of Theorem 4.2.3. Panel (a) shows the case when $\nu^*(t)$ is constant and $\lim_{\tau\to\infty} \pi_\varnothing(\tau) = \nu^*$; no finite $\tau_\varnothing$ exists. Panel (b) illustrates profit margins ν_1^* and ν_2^* which satisfy conditions (i) and (ii) of Theorem 4.2.3, and associated average profit functions $\pi_{\varnothing_1}$ and $\pi_{\varnothing_2}$. Only ν_1^* satisfies condition (iii) of Theorem 4.2.3.

$\nu^*(t) \equiv 1$ and $k \equiv 1$. Hence, $\pi_\varnothing(\tau) = 1 + \frac{1}{\tau}$, $\tau > 0$. Panel (b) of Figure 4.4 depicts two cases where assumptions (i) and (ii) of Theorem 4.2.3 are both satisfied. However, only the profit function $\nu_1^*(t)$ satisfies condition (iii). The value of the optimal cycle length is given by the point where ν_1^* (dark yellow) and the associated average profit $\pi_{\varnothing_1}$ (green

[27]In Example 4.2.1, we showed that if $r > 0$ a finite τ^*, however, does exist.

line) intersect.[28] In the case of ν_2^*, condition (iii) is violated: the profit margin ν_2^* is constant on the interval $[0,2]$ and condition (4.33) is not satisfied for $\tau \in [T_0, 2]$; note $0 < T_0 < 2$. Moreover, the plot of ν_2^* exhibits a jump at $t = 2$ (essentially due to a jump of the μ function). Nevertheless, the average profit function peaks exactly at the same point. The kink of $\pi_{\varnothing 2}(\tau)$ at the optimal point might cause difficulties when doing numerical calculations.

We summarize structural properties of the average-optimal cycle length $\tau_\varnothing$ in the following lemma. Afterwards, we analyze how advertising influences $\tau_\varnothing$. To this end, notice that multiplying the first order condition (4.34) by τ one obtains

$$\pi_1^*(\tau) - k = \tau\nu^*(\tau) = \nu^*(\tau)\int_0^\tau dt = \int_0^\tau \nu^*(\tau)dt.$$

Thus, assuming condition (i) to (iii) of Theorem 4.2.3 to hold, the average-optimal cycle length is the unique solution of

$$\int_0^\tau \left(\nu^*(t) - \nu^*(\tau)\right) dt = k. \tag{4.35}$$

Since $\nu^*(t) > 0$ is monotonically decreasing on $(T_0, +\infty)$, the integrand is nonnegative, and the integral on the left-hand side of (4.35) increases in τ on $(T_0, +\infty)$.[29] Since $\tau_\varnothing$ is the unique solution of (4.35) and the left-hand side of (4.35) is monotone increasing, the optimal cycle length $\tau_\varnothing$ obviously increases in the setup cost k; it takes longer to compensate for the fixed cost k. This observation goes in line with statement 2.(c) of Corollary 4.2.1. Moreover, equation (4.35) enables us to compare the average-optimal cycle lengths for different profit rates. In particular, we are interested in the effect of advertising, i.e., the comparison of the pure dynamic pricing model of Rajan et al. (1992) and our dynamic pricing and dynamic advertising model with respect to the average-optimal cycle length. First, we specify conditions (as part of the following Lemma), that make it possible to compare both models.

[28] For simplicity, we concentrate on the expression for the profit rate and neglect the particular choice of the parameter values that enter the profit margins.

[29] Notice that the derivative of $\int_0^\tau \left(\nu^*(t) - \nu^*(\tau)\right) dt = \int_0^\tau \nu^*(t)dt - \nu^*(\tau)\tau$ with respect to τ is given by $\nu^*(\tau) - \left(\dot\nu^*(\tau)\tau + \nu^*(\tau)\right) = -\dot\nu^*(\tau)\tau$. Since $\nu^*(t)$ monotonically decreases on $(T_0, +\infty)$ the integral $\int_0^\tau \left(\nu^*(t) - \nu^*(\tau)\right) dt$ increases in $\tau > T_0$.

Lemma 4.2.1 *Let the functions $f_1(t)$ and $f_2(t)$ be positive and integrable for all $t \geqslant 0$. Let $k > 0$ and define, for $i = 1, 2$,*

$$F_i(\tau) := \frac{1}{\tau}\left(\int_0^{\tau} f_i(t)dt - k\right). \tag{4.36}$$

Let $T_0^{(1)}, T_0^{(2)}$ resp., denote the root of F_1, F_2 resp., and assume $T_0^{(i)}, i = 1, 2,$ to be positive and finite. Let $\tau_i, T_0^{(i)} < \tau_i < +\infty, i = 1, 2,$ denote the unique maximizer of F_i. Assume f_1, f_2 resp., to be strictly decreasing for all $t \in (T_0^{(1)}, +\infty)$, $t \in (T_0^{(2)}, +\infty)$ resp.

Assume the difference $\alpha(t) := f_2(t) - f_1(t)$ to be positive and nonincreasing for all $t \in [0, \tau_1]$. If $\alpha(t)$ is strictly decreasing on some subinterval of $[0, \tau_1]$, then

$$\tau_2 < \tau_1.$$

Proof. We will proof Lemma 4.2.1 by contradiction. Assume $\tau_2 > \tau_1$. Under this assumption, we will show that the first order condition, see below, can not be satisfied. Since F_i is a smooth function in τ, optimality of $\tau_i > 0$ implies that τ_i must satisfy the first order condition, $i = 1, 2$,

$$\frac{dF_i}{d\tau}(\tau_i) = \frac{f_i(\tau_i)\tau_i - \int_0^{\tau_i} f_i(t)dt + k}{\tau_i^2} \stackrel{!}{=} 0.$$

Thus,

$$\int_0^{\tau_i} f_i(t)dt - f_i(\tau_i)\,\tau_i = \int_0^{\tau_i} (f_i(t) - f_i(\tau_i))\,dt = k. \tag{4.37}$$

Since f_i is strictly positive, both integral expressions $\int_0^{\tau_i} f_i(t)dt$ are increasing in $\tau_i > 0$. This property implies $\tau_1 > T_0^{(1)}$ and $\tau_2 > T_0^{(2)}$, since otherwise $F_i(\tau) < 0$ for all $\tau > 0$. Since f_i is strictly decreasing on the open interval $(T_0^{(i)}, +\infty)$ the expression, $i = 1, 2$,

$$\begin{aligned}\int_0^{\tau} (f_i(t) - f_i(\tau))\,dt &= \int_0^{T_0^{(i)}} (f_i(t) - f_i(\tau))\,dt + \int_{T_0^{(i)}}^{\tau} (f_i(t) - f_i(\tau))\,dt \\ &= k - f_i(\tau)T_0^{(i)} + \int_{T_0^{(i)}}^{\tau} (f_i(t) - f_i(\tau))\,dt\end{aligned}$$

strictly increases in $\tau > T_0^{(i)}$. Hence, the value τ_i is the only solution of equation (4.37). W.l.o.g., let us assume that $\tau_2 \geqslant \tau_1$. Since $f_2(t)$ is decreasing on $(T_0^{(2)}, \infty)$, the relationship $\tau_2 \geqslant \tau_1$ implies the inequality $f_2(\tau_2) \leqslant f_2(\tau_1)$. Thus,

$$\int_0^{\tau_1} (f_2(t) - f_2(\tau_1))\, dt \leqslant \int_0^{\tau_2} (f_2(t) - f_2(\tau_2))\, dt = k, \tag{4.38}$$

cf. equation (4.37). However, recall, $f_2(t) = f_1(t) + \alpha(t), 0 \leqslant t \leqslant \tau_1$, since $\alpha(t) := f_2(t) - f_1(t)$,

$$\begin{aligned}
\int_0^{\tau_1} (f_2(t) - f_2(\tau_1))\, dt &= \int_0^{\tau_1} (f_1(t) + \alpha(t) - (f_1(\tau_1) + \alpha(\tau_1)))\, dt \\
&= \int_0^{\tau_1} (f_1(t) - f_1(\tau_1))\, dt + \int_0^{\tau_1} (\alpha(t) - \alpha(\tau_1))\, dt \\
&= k + \int_0^{\tau_1} (\alpha(t) - \alpha(\tau_1))\, dt \\
&> k,
\end{aligned}$$

where, in the last step, we make use of the fact that $\alpha(t)$ is generally nonincreasing and decreases on some subinterval of $[0, \tau_1]$. Hence, the integrand $\alpha(t) - \alpha(\tau_1)$ is positive on some subinterval of $[0, \tau_1]$. Thus, $\int_0^{\tau_1} (\alpha(t) - \alpha(\tau_1))\, dt > 0$ and the hypothesis $\tau_2 \geqslant \tau_1$ leads to a contradiction. Therefore, $\tau_2 < \tau_1$ must hold true. ♦

In the following, we will interpret f_1 and f_2 as profit margins. Hence, the function $F_i(\tau)$ in Lemma 4.2.1 is the average profit associated with the margin f_i and cycle length τ. Assuming the difference α to be positive is equivalent to the statement that the (profit) function f_2 is strictly larger than the (profit) function f_1. In Theorem 4.2.4, we will formulate assumptions that enable us to compare the optimal cycle length τ_R associated with the profit function ν_R in the pure pricing model ($\delta = 0$, cf. Corollary 2.2.2) with the optimal cycle length $\tau_\varnothing$ associated with profit function ν^* in the dynamic pricing and dynamic advertising model. In Corollary 2.2.3, we introduced the function $\beta(t) > 0$ to link the arrival rate μ_R of the pure pricing model with the arrival rate μ of the model where $\delta > 0$; namely, we assume $\mu_R(t) = \beta(t)\mu(t)$.[30] We argue that due to the lack of advertising in the pure pricing model the sales rate comprises an advertising rate of value one; the function β can be interpreted as an exogenously given (costless) advertising rate. If the optimal advertising rate in the model with advertising is sufficiently large, i.e., $w^*(t)^\delta > \frac{\beta(t)}{1-\Delta}$ for any $t \geqslant 0$, then $\nu^*(t) > \nu_R(t)$, cf. Corollary 2.2.3. Under additional assumptions, we are able to compare the optimal cycle lengths

[30] One can think of two markets: one market where advertising has an effect ($\delta > 0$, the dynamic pricing and advertising model, and arrival intensity μ) and another market where advertising has no effect or is forbidden ($\delta \equiv 0$, i.e., a pure pricing model, and the arrival intensity equals μ_R).

of both markets.

Theorem 4.2.4 *Let $k > 0$, $\mu(t) > 0$, and let the assumptions of Theorem 4.2.3 be satisfied. Let $\beta(t) > 0$, and let ν_R be given by formula* (2.26) *of Corollary 2.2.2 with arrival rate $\mu_R(t) := \beta(t)\mu(t), t \geqslant 0$, ceteris paribus. Assume that*

$$\pi_R(\tau) := \frac{1}{\tau}\left(\int_0^\tau \nu_R(t)dt - k\right) \tag{4.39}$$

has a unique maximizer τ_R and let the optimal value $\pi_R(\tau_R)$ be positive. If, for all $t \in [0, \tau_R]$,

$$w^*(t)^\delta > \frac{\beta(t)}{1-\Delta} \tag{4.40}$$

and $w^*(t)$ is nonincreasing on $[0, T_0]$, see equation (4.6) *for the definition of T_0, then*

$$\tau_\varnothing < \tau_R, \tag{4.41}$$

and

$$\pi_\varnothing(\tau_\varnothing) > \pi_R(\tau_R). \tag{4.42}$$

Proof. Recall, if $\delta > 0$, the optimal profit rate and the optimal advertising rate are given by, see Theorem 2.2.1,

$$\begin{aligned} \nu^*(t) &= \tfrac{c_\lambda}{\gamma-1}\left(\tfrac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{1-\Delta}} &&= \frac{1-\Delta}{\Delta}w^*(t)^a, \\ w^*(t) &= c_w\left(\tfrac{\mu(t)}{c(t)^{\varepsilon-1}}\right)^{\frac{1}{a-\delta}} &&> 0, \end{aligned} \tag{4.43}$$

where $\gamma - 1 = \frac{\varepsilon-1}{1-\Delta}$, $c_w = \left[\frac{\Delta}{\varepsilon-1}\left(\frac{\varepsilon-1}{\varepsilon}\right)^\varepsilon\right]^{\frac{1}{a-\delta}}$, and $c_\lambda = \frac{\varepsilon-1}{\Delta}{c_w}^a$ are positive constants.

If $\delta \equiv 0$, see Corollary 2.2.2, the optimal profit rate is given by

$$\nu_R(t) = \frac{1}{\varepsilon-1}\left(\frac{\varepsilon-1}{\varepsilon}\right)^\varepsilon \frac{\mu_R(t)}{c(t)^{\varepsilon-1}}.$$

Since, by assumption, $w^*(t)^\delta > \frac{\beta(t)}{1-\Delta}$ on $[0, \tau_R]$, according to identity (2.30),

$$\nu^*(t) = \frac{1-\Delta(t)}{\beta(t)}w^*(t)^{\delta(t)}\nu_R(t),$$

it follows that $\nu^*(t) > \nu_R(t)$ on $[0, \tau_R]$. To verify relation (4.41), we will show that the assumptions of Lemma 4.2.1 are satisfied if $f_1 := \nu_R$ and $f_2 := \nu^*$; the associated average-optimal cycle lengths are denoted by $\tau_1 := \tau_R$ and $\tau_2 := \tau_\varnothing$. In particular, we have to show that

(a) $\nu^*(t)$ and $\nu_R(t)$ are decreasing functions for sufficiently large values of t, see Lemma 4.2.1 and further details below,

(b) the difference $\alpha(t) := \nu^*(t) - \nu_R(t)$ is positive and is nonincreasing on $[0, \tau_R]$,

(c) $\alpha(t)$ is strictly decreasing on some subinterval of $[0, \tau_R]$.

By assumption, the optimal cycle length τ_R exists and $\pi_R(\tau_R)$ is positive. By Theorem 4.2.3 the optimal cycle length $\tau_\varnothing$ also exists. Hence, finite minimal cycle lengths T_0 and T_R exist, where $T_R := \inf \left\{ T \,\middle|\, \int_0^T \nu_R(t)dt \geqslant k, T \geqslant 0 \right\} < \tau_R$. Since, by assumption, $\nu^*(t) > \nu_R(t)$ for all $t \geqslant 0$, the inequality $T_0 < T_R$ follows.

(a) We will show that the function $\nu^*(t)$, $\nu_R(t)$ respectively, is decreasing for all $t > T_0$, $t > T_R$ respectively. Since we assume that all conditions of Theorem 4.2.3 are satisfied, $\nu^*(t)$ strictly decreases for all $t > T_0$, see (4.33). Hence, by equation (4.43), also $w^*(t)^a$ decreases for all $t > T_0$. Recall, $w^*(t)$ is positive whenever t and a are positive. Thus, $w^*(t)$ also decreases for all $t > T_0$. Rearranging equation (2.33), $w^*(t)^{a(t)-\delta(t)} = \frac{\Delta(t)}{\beta(t)} e^{R(t)} \nu_R(t)$, one obtains $\nu_R(t) = \frac{\beta}{\Delta} w^*(t)^{a-\delta}$. Since $a > \delta$, the property that the advertising rate is monotonically decreasing on (T_0, ∞) implies that $\nu_R(t)$ also decreases monotonically for all $t > T_0$. Since $T_0 < T_R$, the function ν_R is monotonically decreasing on (T_R, ∞).

(b) & (c) By assumption, ν^* dominates ν_R on $[0, \tau_R]$, a fact which implies that the difference $\alpha(t) = \nu^*(t) - \nu_R(t)$ is positive on $[0, \tau_R]$. Exploiting the identity (2.30), $\nu^*(t) = \frac{1-\Delta(t)}{\beta(t)} w^*(t)^{\delta(t)} \nu_R(t)$, we obtain

$$\alpha(t) = \nu^*(t) - \nu_R(t) = \nu^*(t) - \frac{\beta(t)}{1-\Delta} \frac{\nu^*(t)}{w^*(t)^\delta} = \nu^*(t) \left(1 - \frac{\beta(t)}{1-\Delta} \frac{1}{w^*(t)^\delta} \right). \tag{4.44}$$

Since both factors of the product are positive decreasing functions in t, $\alpha(t)$ also decreases on $(T_0, \tau_R]$, which verifies part (c).

To finish part (b), we will show that $\alpha(t)$ is nonincreasing on $[0, T_0]$. By assumption, the optimal advertising rate w^* does not increase on $[0, T_0]$. Thus, ν^* does not increase on $[0, T_0]$, see (4.43). Hence, $\alpha(t)$ is nonincreasing on $[0, T_0]$, cf. equation (4.44). Thus, all assumptions of Lemma 4.2.1 are satisfied and property (4.41) follows.

Since $\nu^*(t) > \nu_R(t)$ for all $t \geqslant 0$, $\tau_\varnothing < \tau_R$, the following strict inequality,

$$\pi_\varnothing(\tau_\varnothing) = \frac{\int_0^{\tau_\varnothing} \nu^*(t)dt - k}{\tau_\varnothing} \geqslant \frac{\int_0^{\tau_R} \nu^*(t)dt - k}{\tau_R} > \frac{\int_0^{\tau_R} \nu_R(t)dt - k}{\tau_R} = \pi_R(\tau_R),$$

implies (4.42). Note, by assumption, $\tau_\varnothing$ is the average-optimal cycle length maximizing $\pi_\varnothing(\tau)$, thus, the first inequality $\frac{\int_0^{\tau_\varnothing} \nu^*(t)dt-k}{\tau_\varnothing} \geqslant \frac{\int_0^{\tau_R} \nu^*(t)dt-k}{\tau_R}$ follows. ♦

Theorem 4.2.4 allows us to compare the two different models with respect to the average-optimal cycle lengths and the associated average profits per unit time. Equation (2.30) in Corollary 2.2.3,

$$\nu^*(t) = \frac{1-\Delta(t)}{\beta(t)} w^*(t)^{\delta(t)} \nu_R(t),$$

makes it possible to compare the profit rates of the following two scenarios: the model *with* advertising ($\delta > 0$), and the model *without* advertising ($\delta = 0$). Verifying part (*b*)

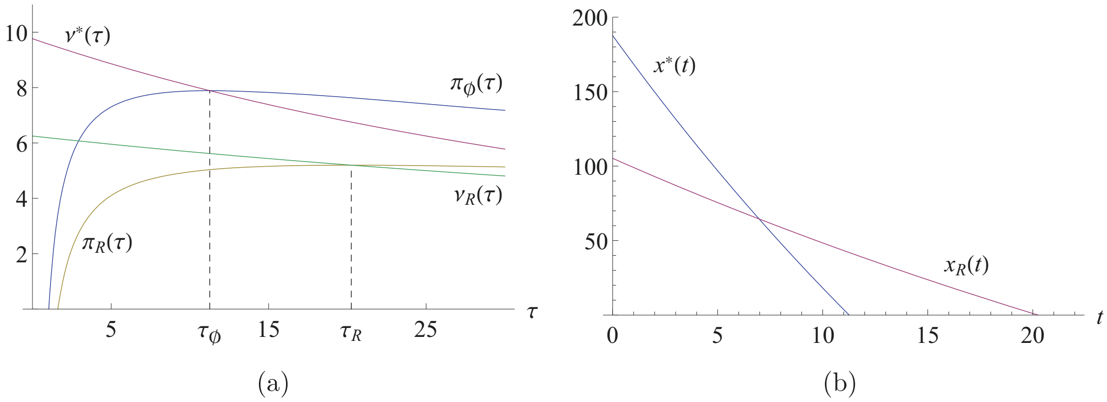


Figure 4.5: Panel (a) shows the average profits and optimal margins as functions of the cycle length τ for the model with advertising ($\delta > 0, \nu^*, \pi_\varnothing$) and the model without advertising ($\delta = 0, \nu_R, \pi_R$). Panel (b) shows the optimally controlled inventory processes for one cycle. The parameters are: $a = 2, \delta = 1, \varepsilon = 2, c_0 = 1, \ell = 0.01, k = 10, q = 0, \mu = \mu_R = 25 (\beta = 1)$.

of the proof of Theorem 4.2.4 is straight forward. For instance, if the arrival intensities are identical for both scenarios, i.e., $\beta = 1$, and $\mu(t) = \mu_R(t)$ for all $t \geqslant 0$, then the opportunity to advertise is profitable[31] only if the optimal advertising rate is larger than one (sales are boosted). Accordingly, if $w^* < 1$, it is optimal to curb demand, and the profit of the scenario with $\delta = 0$ is larger than with $\delta > 0$.[32] It is intuitive to assume $\beta < 1$ because generally there is no *free* promotion. Then, also advertising

[31] *Profitable* here means that the (optimal) profit margin associated with $\delta > 0$ is greater than the (optimal) profit margin associated with $\delta \equiv 0$ on $[0, \tau_R]$, i.e., $\nu^*(t) > \nu_R(t), 0 \leqslant t \leqslant \tau_R$.

[32] This property goes in line with the discussion around Corollary 2.2.3, cf. Chapter 2, when does the opportunity to advertise become an obligation.

rates less than one might satisfy $w^*(t)^\delta > \frac{\beta(t)}{1-\Delta}$.[33] Panel (a) of Figure 4.5 illustrates the optimality conditions for $\tau_\varnothing$ and τ_R if $\beta = 1$ and $q(t) \equiv 0$. Panel (b) on the right depicts the optimally controlled inventory processes for one cycle. One's intuition suggests that, if $\tau_\varnothing < \tau_R$, the initial inventory level if advertising is used is larger than without advertising, i.e., $x^*(0) > x_R(0)$: selling the goods at a higher (demand) rate and at a higher profit rate should also convince the monopolist to sell more items.

If the assumptions of Theorem 4.2.4 are not satisfied, it is not clear which cycle length will be the larger one. For instance, if inequality (4.40) does not hold true for all $t \geqslant 0$, i.e., advertising is not profitable on $[0, \tau_R]$, then it might be possible that $\tau_\varnothing > \tau_R$. For example, setting $\varepsilon = 3$ (instead of $\varepsilon = 2$) in the example of Figure 4.5 leads to a negative value of the difference $\alpha(t)$ for all $t \geqslant 0$, but the inequality $\tau_\varnothing \approx 15.25 < 19.66 \approx \tau_R$ is still satisfied. However, the inequality (4.42) changes, i.e., $\pi_\varnothing(\tau_\varnothing) < \pi_R(\tau_R)$.[34] If $\varepsilon = 4$ - ceteris paribus - the optimal cycle length of the model without advertising is smaller than the optimal cycle length of the model associated with customers who are sensitive to advertising; to be specific, $\tau_\varnothing \approx 21.47 > 20.31 \approx \tau_R$. In general, if the assumptions of Theorem 4.2.4 are not satisfied, then the only way to compare the two models, i.e., the one with advertising and the one without advertising, is by numerical analysis.

4.3 Optimal Order Decisions with Revenue Maximizing Pricing-Advertising Policies

In the previous Section 4.2, we determined the *profit* maximizing policies of a firm which is allowed to simultaneously choose the initial capacity, the duration of the inventory cycle as well as the dynamic pricing and advertising policies for the class of inventory dynamics where no explicit effects depending on the stock level are taken into account. For the class of models with (state) feedback considered in Chapter 3 we propose the following (suboptimal) procedure for a decision maker who decides on the order quantity x and the cycle length τ and who takes the revenue and all inventory related costs of a fixed (still depending on (τ, x)) strategy pair into account. For any pair (τ, x) and for the sales rate

$$\lambda(t, p, w, y) = \mu(t) p^{-\varepsilon} w^\delta \psi(y), \tag{4.45}$$

$0 \leqslant t \leqslant \tau, 0 \leqslant y \leqslant x$, cf. (3.4), the firm chooses (i) the *revenue* (minus advertising costs) *maximizing* policies $(p^\star(t), w^\star(t))$ determined in Chapter 3, see Theorem 3.2.1, and (ii)

[33] Recall, an advertising rate of one might be interpreted in terms of one thousand (dollars).
[34] Notice that in this case $\nu_R(t) > \nu^*(t)$ for all $t \geqslant 0$.

computes the associated costs as follows:

$$k + c_0 x + \int_0^{\tau} e^{-rt} \ell(t) x^{\star}(t,\tau,x) dt,$$

where $x^{\star}(t,\tau,x) := x^{\star}(t)$ is the associated *revenue maximizing* inventory trajectory. This trajectory is given by formula (3.23) where, on the right-hand side of (3.23), T is replaced by τ and x_0 is replaced by x. Thus, considering the average profit per time unit the decision maker faces the following "newsvendor-like" problem:

$$\sup_{\tau,x>0} \left\{ \frac{1}{\tau} \Big[V(\tau,x) - (c_0 x + L(\tau,x) + k) \Big] \right\}. \tag{4.46}$$

$V(\tau,x)$ is the optimal revenue minus the advertising cost when the initial inventory value x is sold over a cycle of length τ, cf. equation (3.51) in Theorem 3.2.2, and

$$L(\tau,x) := \int_0^{\tau} e^{-rs} \ell(s) x^{\star}(s,\tau,x) ds \tag{4.47}$$

is the present value of total inventory costs associated with a pair (τ,x) and the revenue maximizing policy. Besides the time-average problem (4.46) we are also going to analyze the problem where the decision maker wants to maximize the present value of $N \geqslant 1$ inventory cycles while applying the (identical) revenue maximizing policies within each cycle:

$$\sup_{\tau,x>0} \left\{ S_N(\tau) \Big[V(\tau,x) - (c_0 x + L(\tau,x) + k) \Big] \right\}. \tag{4.48}$$

Since $S(T) = \lim_{N\to\infty} S_N(T)$ is well defined, cf. Proposition 4.2.1, we will also consider the case when the number of cycles is infinite. Like in Section 4.2, we will first consider the N cycle problem (4.48) and then the maximization of the time-average profit (4.46).

Note, whenever $\psi(x) \equiv 1$, the results of Chapter 3 imply that for any pair (τ,x) the revenue maximizing strategies $(p^{\star}(t), w^{\star}(t))$ are admissible policies for the problem considered in Chapter 2 and in Section 4.2. Therefore, a solution of (4.46), resp. (4.48), will be suboptimal compared to the solutions derived earlier (in the case $\psi(x) \equiv 1$). Nevertheless, as will be shown below, in many applied cases the solutions of (4.46) and (4.48) will lead to reasonable decision rules.

To solve (4.48), we shall first analyze the objective function without taking the constant k and the terms $1/\tau$ and $S_N(\tau)$ into account. To this end, let

$$\Pi_1^\star(\tau, x) := V(\tau, x) - c_0 x - L(\tau, x). \tag{4.49}$$

The subscript "$_1$" refers to one cycle and the superscript "*" indicates that the revenue maximizing policy is applied. Since $V(\tau, x) = A(t)^{\frac{1}{\gamma}} B(x)^{\frac{\gamma-1}{\gamma}}$ is a separable function of τ and x, cf. (3.51), the function $\Pi_1^\star$ has an appealing economic interpretation. To begin with, for every order scheme (τ, x) the revenue $V(\tau, x)$ is positive and increasing. The special form of V suggests the idea to think of V as a production function, and to think of the variables τ and x as the production factors "time" and "quantity". The output on a value basis associated with a pair (τ, x) is given by $V(\tau, x)$. Then, since the exponents $\frac{1}{\gamma}$ and $\frac{\gamma-1}{\gamma}$ add-up to one, V is a Cobb-Douglas function which is homogeneous of degree 1. The expressions $A(\tau)$ and $B(x)$ are but transformed values of (storage/production) time and (storage/production) capacity; but any other interpretation of a two factor model is also feasible. The costs of using the two production factors are given as a sum of a linear expression of one of the factors, viz. $c_0 \cdot x$, and a combined cost term given by $L(\tau, x)$.

The next three propositions are collections of formulas to be exploited in the sequel. All formulas except (4.54) follow from elementary calculus.

Proposition 4.3.1 *Let Condition 4.1.1 and all hypotheses of Theorem 3.2.1 be satisfied. Let $r(t) = r \geqslant 0$ and let the arrival intensity $\mu(t)$, see (4.45), be differentiable. Then $\Pi_1^\star(\tau, x)$ is a differentiable function, where*

$$\dot{\Pi}_1^\star(\tau, x) := \frac{\partial \Pi_1^\star}{\partial \tau}(\tau, x) = \dot{V}(\tau, x) - \dot{L}(\tau, x), \tag{4.50}$$

$$\Pi_1^{\star\prime}(\tau, x) := \frac{\partial \Pi_1^\star}{\partial x}(\tau, x) = V'(\tau, x) - c_0 - L'(\tau, x). \tag{4.51}$$

Moreover,

$$\dot{V}(\tau, x) := \frac{\partial V}{\partial \tau}(\tau, x) = \frac{e^{-\gamma r \tau}\eta(\tau)}{\gamma A(0, \tau)} V(\tau, x) > 0, \tag{4.52}$$

$$V'(\tau, x) := \frac{\partial V}{\partial x}(\tau, x) = \frac{\psi(x)^{\frac{1}{\varepsilon-1}}}{B(x)} V(\tau, x) > 0, \tag{4.53}$$

$$\dot{L}(\tau, x) := \frac{\partial L}{\partial \tau}(\tau, x) = \int_0^\tau e^{-rt}\ell(t)\frac{\partial x^\star}{\partial \tau}(t, \tau, x)dt > 0, \tag{4.54}$$

$$L'(\tau, x) := \frac{\partial L}{\partial x}(\tau, x) = \int_0^\tau e^{-rt}\ell(t)\frac{\partial x^\star}{\partial x}(t, \tau, x)dt > 0. \tag{4.55}$$

Proof. To verify (4.54) apply Leibniz's formula to the integral expression (4.47) and use the fact that $x^\star(\tau,\tau,x)=0$. Recall, the revenue maximizing strategies $(p^\star(t),w^\star(t))$ guarantee that any capacity x will be sold over any time interval τ. The positivity of expressions (4.52) to (4.55) follows from the assumptions of Theorem 3.2.1 and Condition 4.1.1. ♦

Besides expressions for the derivatives of $\Pi_1^\star$, we also need derivative formulas of the optimal state process $x^\star$ with respect to the time horizon τ and the capacity x, cf. (4.54) and (4.55).

Proposition 4.3.2 *Let all assumptions of Proposition 4.3.1 be satisfied. For any t, $0<t<\tau$, and x, $x>0$,*

$$\frac{\partial x^\star}{\partial \tau}(t,\tau,x) = \frac{e^{-\gamma r\tau}\eta(\tau)}{A^{(0)}(0,\tau)}\left(1-\frac{A^{(0)}(t,\tau)}{A^{(0)}(0,\tau)}\right)\frac{B(x)}{B'(x^\star(t,\tau,x))} > 0, \tag{4.56}$$

$$\frac{\partial x^\star}{\partial x}(t,\tau,x) = \frac{A^{(0)}(t,\tau)}{A^{(0)}(0,\tau)}\frac{B'(x)}{B'(x^\star(t,\tau,x))} > 0. \tag{4.57}$$

Proof. We shall spell out the details of the derivation of equation (4.56). Formula (4.57) can be verified along the same lines. Note, $\left(B^{-1}\right)'(x)=\frac{1}{B'(B^{-1}(x))}$ and the derivative of $A^{(0)}(t,\tau)$ with respect to τ does not depend on t; $\frac{\partial A^{(0)}(t,\tau)}{\partial\tau}=e^{-\gamma r\tau}\eta(\tau)$. Hence,

$$\begin{aligned}\frac{\partial x^\star}{\partial \tau}(t,\tau,x) &= \partial\left(B^{-1}\left(B(x)\frac{A^{(0)}(t,\tau)}{A^{(0)}(0,\tau)}\right)\right)/\partial\tau \\ &= \frac{1}{B'\left(B^{-1}\left(B(x)\frac{A^{(0)}(t,\tau)}{A^{(0)}(0,\tau)}\right)\right)}\frac{B(x)\left[\frac{\partial A^{(0)}}{\partial\tau}(t,\tau)A^{(0)}(0,\tau)-A^{(0)}(t,\tau)\frac{\partial A^{(0)}}{\partial\tau}(0,\tau)\right]}{A^{(0)}(0,\tau)^2} \\ &= \frac{B(x)}{B'\left(x^\star(t,\tau,x)\right)}\frac{\frac{\partial A^{(0)}}{\partial\tau}(t,\tau)A^{(0)}(0,\tau)\left[1-\frac{A^{(0)}(t,\tau)}{A^{(0)}(0,\tau)}\right]}{A^{(0)}(0,\tau)^2},\end{aligned}$$

and equation (4.56) follows. ♦

While the combination of both propositions yields nice explicit formulas, evaluating these expressions in specific cases is easier said than done. To compute the inventory costs $L(\tau,x)$ and their derivative. one has to compute the optimal trajectory $x^\star(t)$ and evaluate various integrals along this path. Therefore, when illustrating some general results later on we shall choose parameter values in such a way that $x^\star(t)$ becomes a *nice* expression, for example, a linear function of t.

Our ultimate goal is to solve the problems (4.46) and (4.48). To this end, we take a closer look at the domain of $\Pi_1^\star$. Obviously, an order scheme (τ, x) should be such that the setup cost $k > 0$ will be covered in every cycle. Therefore, we restrict the domain of $\Pi_1^\star$ to the set $\Lambda_k := \{(\tau, x) | \Pi_1^\star(\tau, x) \geqslant k, \tau > 0, x > 0\}$. For the remainder of the section we assume $\Lambda_k \neq \varnothing$; if no such order scheme exists, it is not profitable to run the business at all. Similar to the minimum cycle length T_0, cf. (4.6), we define the minimum (capacity-dependent) cycle length τ_0 as, $x_0 > 0$,

$$\tau_0(x_0) := \inf \{T \,|\Pi_1^\star(T, x_0) \geqslant k, T > 0\}. \tag{4.58}$$

For a given capacity x_0, the value $\tau_0(x_0)$ is the shortest duration of a cycle that is necessary to make a profit of at least k. If no such value exists, we set $\tau_0(x_0) = +\infty$. Similarly, we define the minimum (cycle length dependent) capacity χ_0, $T > 0$,

$$\chi_0(T) = \inf \{x \,|\Pi_1^\star(T, x) \geqslant k, x > 0\}. \tag{4.59}$$

For a given cycle length T, the value $\chi_0(T)$ denotes the minimum capacity that a firm must establish to break even if all costs, including the setup cost k, are taken into account. If no such value exists, we set $\chi_0(T) = +\infty$. From now on, we will refer to the existence of finite values $\tau_0(x_0)$ or $\chi_0(T)$ as the *market-entry condition*. The market-entry condition guarantees that the set Λ_k is non-empty.

Like for the problems considered in Section 4.2, we provide an integral representation of the function $\Pi_1^\star$ in terms of a profit rate function. The details of the representation are given in the following proposition.

Proposition 4.3.3 *Let all assumptions of Theorem 3.2.1 be satisfied. In equation* (3.25)*, replace T by τ and x_0 by x, and let $w^\star(t) =: w^\star(t, \tau, x)$. Define*

$$\nu^\star(t, \tau, x) := e^{-rt} \left(\frac{\varepsilon - \Delta}{\Delta} w^\star(t, \tau, x)^a - \ell(t) x^\star(t, \tau, x) \right) - c_0 \frac{x}{\tau}. \tag{4.60}$$

Then, the function $\Pi_1^\star(\tau, x)$ defined by (4.49) *has the integral representation*

$$\Pi_1^\star(\tau, x) = \int_0^\tau \nu^\star(t, \tau, x) dt. \tag{4.61}$$

Proof. We exploit the (pointwise) Dorfman-Steiner relation (3.17) as follows:

$$\begin{aligned} V(\tau,x) &= \int\limits_0^\tau e^{-rt}\left(p^\star(t)\lambda^\star(t) - w^\star(t)^a\right)dt \\ &= \int\limits_0^\tau e^{-rt}\left(\frac{\varepsilon}{\Delta}w^\star(t)^a - w^\star(t)^a\right)dt \\ &= \int\limits_0^\tau e^{-rt}\frac{\varepsilon-\Delta}{\Delta}w^\star(t)^a dt. \end{aligned}$$

Hence, using definition (4.60),

$$\begin{aligned} \Pi_1^\star(\tau,x) &= V(\tau,x) - c_0x - L(\tau,x) \\ &= \int\limits_0^\tau e^{-rt}\frac{\varepsilon-\Delta}{\Delta}w^\star(t)^a dt - \frac{c_0x}{\tau}\int_0^\tau dt - \int_0^\tau e^{-rt}\ell(t)x^\star(t,\tau,x)dt \\ &= \int_0^\tau \left[e^{-rt}\left(\frac{\varepsilon-\Delta}{\Delta}w^\star(t,\tau,x)^a - \ell(t)x^\star(t,\tau,x)\right) - c_0\frac{x}{\tau}\right]dt \\ &= \int_0^\tau \nu^\star(t,\tau,x)dt. \end{aligned}$$

♦

The integral representation (4.61) is useful to derive properties of $\Pi_1^\star(\tau,x)$. For instance, the representation reveals how the profit value $\Pi_1^\star(\tau,x)$ depends on the order scheme (τ,x). Specifically, the integrand $\nu^\star(t,\tau,x)$ reveals when profits can be reaped during the business cycle and when losses will be incurred. There are, however, crucial differences between the profit rate $\nu^*(t)$ considered in Section 4.2 and the profit rate $\nu^\star(t,\tau,x)$ given by (4.60). The former does not explicitly depend on (τ,x); only the domain of $\nu^*(t)$ is determined by τ. In Section 4.2, the fact that $\nu^*(t)$ does not explicitly depend on (τ,x) has been exploited several times, for example in Theorems 4.2.1, 4.2.2, and 4.2.3. Since the rate $\nu^\star(t,\tau,x)$ explicitly depends on (τ,x) we can not apply these theorems in the context of the profit rate $\nu^\star(t,\tau,x)$. Moreover, in Section 4.2, all but the setup cost k is considered when the optimal controls are derived. Hence, $\nu^*(t)$ is always positive and thus $\pi_1^*(\tau) = \int_0^\tau \nu^*(t)dt > k$ for *all* $\tau > T_0$; recall, T_0 is the minimum cycle length that guarantees the one-cycle profit to be at least k. In this section, however, a finite value $\tau_0(x_0)$ does *not* guarantee that the one-cycle profit $\Pi_1^\star(\tau,x_0)$ is larger than the setup cost k for all $\tau > \tau_0(x_0)$: if τ becomes large, the revenue rate from applying the (revenue maximizing) policies $(p^\star(t), w^\star(t))$ might be dominated by the inventory costs $L(\tau, x_0)$.[35] For example, consider a market environment where the arrival intensity $\mu(t)$ increases over time. In such a case, the revenue rate will also be an increasing function of t, and a fairly large portion of the total inventory will be stored for most of the cycle length. The total storage cost $L(\tau, x)$ might be larger than the revenue $V(\tau, x)$.

In the following Subsection 4.3.1, we study problem (4.48), where the objective of the firm is to choose an order scheme that maximizes the present value of N inventory cycles. Thereafter, in Subsection 4.3.2, we consider the maximization of the average profit per time unit (4.46).

Quite often, a decision maker does not have the freedom to choose both values τ and x independently. For example, if the capacity of a warehouse is fixed, only the decision on the replenishment intervals remains to be dealt with. On the other hand, if a supplier presets the delivery dates, the retailer only has to choose the quantity to be delivered. In both of the following subsections, we will solve the problem of finding optimal order schemes as follows: we first consider the (one-dimensional) subproblems of finding an optimal cycle length if the capacity is exogenously given, and of finding an optimal capacity if the cycle length is exogenously given. Based on these results, in each subsection, we formulate assumptions that guarantee the existence of an optimal (two-dimensional) *revenue maximizing* order scheme. Finally, in Subsection 4.3.3, we derive the *endogenized Harris-Wilson* formula, see Proposition 4.3.8, and we illustrate some of our results by looking at a stylized example.

4.3.1 Maximizing the Present Value of N Cycles

We assume the number of inventory cycles N to be a fixed positive integer value; for the infinite cycle problem we let $N \to +\infty$. Let $\Pi_N(\tau, x)$ denote the present value of N (identical) inventory cycles of length τ and capacity x, i.e.

$$\Pi_N(\tau, x) := S_N(\tau)\big[\Pi_1^\star(\tau, x) - k\big] = S_N(\tau)\big[V(\tau, x) - (c_0 x + L(\tau, x) + k)\big]. \tag{4.62}$$

We defined the annuity factor S_N in Proposition 4.2.1. Problem (4.48) is equivalent to the problem of finding an order scheme $(\tau_N^\star, x_N^\star) \in \Lambda_k$ that satisfies $\Pi_N(\tau_N^\star, x_N^\star) \geqslant \Pi_N(\tau, x)$ for all feasible order schemes. We will call any such pair $(\tau_N^\star, x_N^\star)$ an optimal order scheme of the N-cycle problem (4.48). Recall, any order scheme which is an

[35] An even simpler example where the total net profit becomes negative is when the unit cost c_0 are *too* high. In general, we assume that the maximized revenue exceeds the total unit cost, i.e., $V(\tau, x) > c_0 x$, and that the running and the setup costs are the cost factors which possibly let the one-cycle profit become negative.

element of the set (of feasible order schemes) Λ_k guarantees nonnegative profits over one cycle; by definition, $\Pi_1^\star(\tau, x) - k \geqslant 0$ for all $(\tau, x) \in \Lambda_k$.

In the sequel, to identify an optimal order scheme we will first consider the family of one-dimensional subproblems of maximizing $\Pi_N(\tau, x_0)$ with respect to τ, $x_0 > 0$ arbitrary but fixed. In a second step, we consider the family of problems of maximizing $\Pi_N(T, x)$ with respect to x, $T > 0$ arbitrary but fixed. The next proposition specifies a condition such that a x_0-subproblem has a solution.

Proposition 4.3.4 *Assume all conditions of Proposition 4.3.1 hold true. Let $x_0 > 0$ be arbitrary but fixed, and let the minimum cycle length $\tau_0(x_0)$ exist, cf.* (4.58).
Whenever there is a finite value $\bar{\tau}_N := \bar{\tau}_N(x_0)$, $\bar{\tau}_N \geqslant \tau_0(x_0)$, such that $\Pi_N(\bar{\tau}_N, x_0) \geqslant \Pi_N(\tau, x_0)$ for all $\tau > \bar{\tau}_N$, then a maximizing cycle length $\tau_N^\star(x_0)$, $\tau_0(x_0) \leqslant \tau_N^\star(x_0) \leqslant \bar{\tau}_N(x_0)$, exists. Moreover, $\tau_N^\star(x_0)$ solves the equation

$$\Pi_1^\star(\tau, x_0) - k = -\frac{S_N(\tau)}{\dot{S}_N(\tau)} \dot{\Pi}_1^\star(\tau, x_0). \tag{4.63}$$

Proof. Since $\Pi_1^\star(\tau, x_0)$ is differentiable in τ, cf. Proposition 4.3.1, and $S_N(\tau)$ is differentiable as well, the product $\Pi_N(\tau, x_0) = S_N(\tau)\,[\Pi_1^\star(\tau, x_0) - k]$ is differentiable in τ. Moreover, $\Pi_N(\tau, x_0)$ is negative on $(0, \tau_0(x_0))$; recall, $\tau_0(x_0)$ is the smallest value such that $\Pi_1^\star(\tau_0(x_0), x_0) - k$ is nonnegative. Furthermore, by assumption, the function $\Pi_N(\tau, x_0)$ is bounded from above by the value $\Pi_N(\bar{\tau}_N, x_0)$ on $(\bar{\tau}_N, +\infty)$. Hence, the continuous function $\Pi_N(\tau, x_0)$ attains its global maximum in the interior of a compact interval which contains the interval $[\tau_0(x_0), \bar{\tau}_N(x_0)]$. Since $\Pi_N(\tau, x_0)$ is differentiable in τ, the first order condition

$$\frac{\partial \Pi_N}{\partial \tau}(\tau, x_0) = \dot{S}_N(\tau)\,[\Pi_1^\star(\tau, x) - k] - S_N(\tau)\dot{\Pi}_1^\star(\tau, x) = 0$$

is satisfied for $\tau_N^\star(x_0)$. Rewriting the first order condition yields equation (4.63). ♦

Remark 4.3.1 *We do not require the optimal value $\Pi_N(\tau_N^\star(x_0), x_0)$ to be strictly positive. In case the optimal value is zero, the monopolist is indifferent between entering the market and running no business. To make sure that the optimal value is strictly positive, it is sufficient to find some point where the function $\Pi_N(\tau, x_0)$ is positive. Note, the value of Π_N at the end point $\bar{\tau}_N = \bar{\tau}_N(x_0)$ is allowed to be positive or negative. To find such a point $\bar{\tau}_N$, it is sufficient to find such a point for the function $\Pi_1^\star(\tau, x_0)$. However, the condition imposed on the function $\Pi_N(\tau, x_0)$ is general since $\Pi_N(\tau, x_0)$ involves the annuity factor $S_N(\tau)$.*

The existence of an optimal cycle length (depending on x_0) can be guaranteed if the assumptions of Proposition 4.3.4 are satisfied. The assumptions, however, do not imply uniqueness. For example, if $\Pi_N(\tau, x_0)$ is a unimodal function[36] on the interval $(\tau_0(x_0), \bar{\tau}_N(x_0))$, then $\tau_N^\star(x_0)$ is unique. For a given collection of parameters and a given value x_0, the optimal cycle length can - and typically needs to be - determined numerically. If $\Pi_N(\tau, x_0)$ is unimodal (in τ), such elementary procedures as the golden section search can be used to compute $\tau_N^\star(x_0)$.

In a similar way as in Proposition 4.3.4, we formulate conditions that guarantee the existence of an optimal capacity value $x^\star(T)$ whenever the cycle length T is given.[37] The proof of the following result runs along the same lines as the proof of Proposition 4.3.4, and is therefore omitted. Actually, the proof is slightly simpler than the one of Proposition 4.3.4 since the factor $S_N(T)$ only depends on N and T but not on the capacity value x_0.

Proposition 4.3.5 *Assume all conditions of Proposition 4.3.1 hold true. Let $T > 0$ be arbitrary but fixed, and let the minimum capacity $\chi_0(T)$ exist, cf.* (4.59).

Whenever there is a finite value $\bar{\chi} := \bar{\chi}(T)$, $\bar{\chi} \geqslant \chi_0(T)$, such that $\Pi_1^\star(T, \bar{\chi}) \geqslant \Pi_1^\star(T, x)$ for all $x > \bar{\chi}(T)$, then a maximizing capacity $x^\star(T)$, $\chi_0(T) \leqslant x^\star(T) \leqslant \bar{\chi}(T)$, exists. Moreover, $x^\star(T)$ solves the equation

$$V'(T, x) = c_0 + L'(T, x). \tag{4.64}$$

Remark 4.3.2 *Again, we do not require the optimal value to be strictly positive, i.e., the monopolist might be indifferent between entering the market expecting zero profit and running no business. To ensure that the optimal value is bigger than zero, it is sufficient to find some point where the function $\Pi_N(T, x)$ is positive. The value of Π_N at the end point $\bar{\chi}(T)$ is allowed to be positive or negative. To find such a point $\bar{\chi}(T)$, it is sufficient to concentrate on the one-cycle profit function $\Pi_1^\star(T, x)$ since the factor $S_N(T)$ does not depend on x. Furthermore, the maximizing capacity $x^\star(T)$ does not depend on the number of cycles N.*

Similar to the case when the capacity is assumed to be fixed, additional information about properties of the objective function facilitates this analysis. For example, unimodality (in x) of the function $\Pi_1^\star(T, x)$ is one such useful property. The first order

[36] A function $f(x)$ is a unimodal function if for some value α, it is strictly monotonically increasing for $x \leqslant \alpha$ and strictly monotonically decreasing for $x \geqslant \alpha$. In that case, the maximum value of $f(x)$ is $f(\alpha)$ and there are no other local maxima.

[37] Below, we show that the optimal capacity does not depend on the number of cycles if T is exogenously given. Thus, we omit the subscript $'_N'$ and write $x^\star(T)$.

condition (4.64) reveals another important property of $x^\star(T)$: the optimal capacity value does not depend on the order cost k as long as the market-entry condition is fulfilled, i.e., there is at least one capacity value such that the fixed cost k is covered, or in other words, $\chi_0(T)$ is a finite value. It is clear that $x^\star(T)$ does not depend on k, since for any (positive) order quantity x the fixed amount of order cost has to be paid.

Both propositions, 4.3.4 and 4.3.5, are about one-dimensional optimization problems. The following theorem specifies assumptions that guarantee the existence of an optimal order scheme $(\tau_N^\star, x_N^\star)$ of the two-dimensional optimization problem (4.48).

Theorem 4.3.1 *Assume all conditions of Proposition 4.3.1 hold true. There exists an optimal pair $(\tau_N^\star, x_N^\star)$ of problem (4.48), if any of the following conditions, (i), (ii), or (iii), are satisfied:*

(i) (a) There is a strip $\mathbb{R}_+ \times [x_1, x_2]$, $0 < x_1 < x_2 < \infty$, and a point $(\sigma, y) \in \Lambda_k$, $x_1 < y < x_2$, such that for all $(\tau, x) \in \Lambda_k, x \leqslant x_1$ or $x \geqslant x_2 : \Pi_N(\tau, x) < \Pi_N(\sigma, y)$.

(b) There is a continuous selection function $\tau_N^\star(x)$, $x \in [x_1, x_2]$, of solutions of the one-dimensional problems considered in Proposition 4.3.4.

(ii) (a) There is a strip $[T_1, T_2] \times \mathbb{R}_+$, $0 < T_1 < T_2 < \infty$, and a point $(\sigma, y) \in \Lambda_k$, $T_1 < \sigma < T_2$, such that for all $(\tau, x) \in \Lambda_k, \tau \leqslant T_1$ or $\tau \geqslant T_2 : \Pi_N(\tau, x) < \Pi_N(\sigma, y)$.

(b) There is a continuous selection function $x^\star(T)$, $T \in [T_1, T_2]$, of solutions of the one-dimensional problems considered in Proposition 4.3.5.

(iii) There exists a rectangle $[T_1, T_2] \times [x_1, x_2]$, $0 < T_1 < T_2 < \infty, 0 < x_1 < x_2 < \infty$, and a point $(\sigma, y) \in \Lambda_k$, $T_1 < \sigma < T_2, x_1 < y < x_2$, such that $\Pi_N(\tau, x) < \Pi_N(\sigma, y)$ for all $\tau \notin (T_1, T_2)$ and $x \notin (x_1, x_2)$.

Proof. We prove the existence of an optimal pair $(\tau_N^\star, x_N^\star)$ for each set of conditions (i), (ii), and (iii).

(i) By assumption, there exists a feasible pair (σ, y) such that $\Pi_N(\tau, x) < \Pi_N(\sigma, y)$ for all feasible pairs (τ, x), where $x \notin (x_1, x_2)$. Hence, if a global maximum of the function $\Pi_N(\tau, x)$ exists, this global maximum lies in the interior of the strip $\mathbb{R}_+ \times [x_1, x_2]$; recall that $\Pi_N(0, x) < 0$ for all $x > 0$. Moreover, the function $\Pi_N(\tau, x)$ is continuous in τ and in x (separately). Since, by assumption, a continuous selection function $\tau_N^\star(x), x \in [x_1, x_2]$ exists, the function $\Pi_N(\tau_N^\star(x), x)$ is continuous (in x) on the interval $[x_1, x_2]$. It will attain its global maximum at some point $x_N^\star$ in the interior of the interval $[x_1, x_2]$. Hence, an optimal order scheme $(\tau_N^\star, x_N^\star)$ exists, where $\tau_N^\star := \tau_N^\star(x_N^\star)$.

(*ii*) The proof runs along the same lines as the proof of (*i*); simply change the point of view from the capacity dimension to the time dimension.

(*iii*) By assumption, there exists a feasible pair (σ, y) such that $\Pi_N(\tau, x) < \Pi_N(\sigma, y)$ for all feasible pairs (τ, x) outside or on the boundary of the rectangle $[T_1, T_2] \times [x_1, x_2]$. Hence, if a global maximum of the function $\Pi_N(\tau, x)$ exists, this global maximum lies in the interior of the rectangle. Since the function $\Pi_N(\tau, x)$ is continuous in τ and in x (separately), it will attain its global maximum on $(T_1, T_2) \times (x_1, x_2)$. ♦

Remark 4.3.3 *Conditions* (*i*) *and* (*ii*) *in Theorem 4.3.1 are refined versions of condition* (*iii*). *We shall exploit the refined conditions when analyzing an illustrative example of problem* (4.48)*, see Section 4.3.3.*

The conditions (*i*), (*ii*), and (*iii*) are not only of *technical* nature, but represent distinct approaches to the two-dimensional problem (4.48). For example, if the objective function $\Pi_N(\tau, x)$ reveals properties which make it easy to solve the one-dimensional problem associated with Proposition 4.3.4, then it is viable to work with the set of conditions (*i*) and look for a (regular) selection function $\tau_N^\star(x)$. The set of conditions (*ii*) is useful if the one-dimensional capacity problem is easy to solve. Part (*iii*) of Theorem 4.3.1 states conditions on the two-dimensional problem in a general form, as no selection function in one or the other variable is assumed to exist. However, whenever one has to deal with a specific problem of type (4.48) one is well advised to focus on one of the one-dimensional subproblems, cf. Section 4.3.3. The goal is always to (numerically) determine solutions to the first order conditions in τ and x, see Corollary 4.3.1, and to verify that the sufficient conditions are satisfied. The latter is usually the hard task. Each part of the system of equations (4.65) and (4.66) is identical to the first order condition of either the one-dimensional problem associated with Proposition 4.3.4 or with Proposition 4.3.5.

Corollary 4.3.1 *Assume any of the conditions* (*i*), (*ii*), *or* (*iii*) *of Theorem 4.3.1 to be satisfied. Then, the optimal pair* $(\tau_N^\star, x_N^\star)$ *is a solution of the system of equations*

$$\Pi_1^\star(\tau, x) - k = -\frac{S_N(\tau)}{\dot{S}_N(\tau)} \left[\dot{V}(\tau, x) - \dot{L}(\tau, x) \right], \tag{4.65}$$

$$V'(\tau, x) = c_0 + L'(\tau, x). \tag{4.66}$$

4.3.2 Maximizing the Average Profit per Time Unit

Let $\Pi_{\varnothing}(\tau, x)$ denote the average profit per time unit of an inventory cycle of length τ and inventory capacity x, see the beginning of Section 4.3,

$$\Pi_{\varnothing}(\tau, x) := \frac{1}{\tau}\Big[\Pi_1^{\star}(\tau, x) - k\Big] = \frac{1}{\tau}\Big[V(\tau, x) - (c_0 x + L(\tau, x) + k)\Big]. \tag{4.67}$$

Finding an order scheme $\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right) \in \Lambda_k$ that satisfies $\Pi_{\varnothing}\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right) \geqslant \Pi_{\varnothing}(\tau, x)$ for all feasible order schemes, is equivalent to problem (4.46). Any such pair $\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$ will be called an (average) optimal order scheme of problem (4.46). The set of feasible order schemes Λ_k is identical to the set of feasible order schemes of problem (4.48). This is obvious, since (a) the first factor of the objective function of both problems is strictly positive, i.e., $S_N(\tau) > 0$ and $1/\tau > 0$ for all $\tau > 0$, and (b) each nonnegative one-cycle profit value $\Pi_1^{\star}(\tau, x)$ determines a feasible order scheme; the value $\Pi_1^{\star}$ also guarantees that the present value of N cycles as well as the average profit per time unit is nonnegative.

When the objective is to maximize the average profit per time unit, it is natural to assume no discounting, cf. Section 4.2.2. In this Section, we abstain from explicitly assuming $r \equiv 0$ since the following results remain valid for the *odd* case of discounted average profits. In particular, the expressions $V(\tau, x)$ and $L(\tau, x)$, see (3.51) and (4.47), as well as the derivatives of $V(\tau, x)$ and $L(\tau, x)$ with respect to τ, cf. Proposition 4.3.1, are well defined if $r \equiv 0$. The assumptions and conditions which imply the existence of an order scheme that maximizes the profit per time unit are similar to those in Section 4.3.1. For example, the conditions that guarantee the existence of an optimal capacity of the one-dimensional subproblem when the cycle length T is exogenously given are identical for both objectives, $\Pi_{\varnothing}(T, x)$ and $\Pi_N(T, x)$. This is due to the fact that the factors $S_N(\tau)$ and $1/\tau$ do not depend on the capacity x.

For the remainder of this section we follow the outline of Section 4.3.1: we first consider the family of one-dimensional subproblems of maximizing $\Pi_{\varnothing}(\tau, x_0)$ with respect to τ, where $x_0 > 0$ is arbitrary but fixed. Then, we analyze the family of subproblems of maximizing $\Pi_{\varnothing}(T, x)$ with respect to x, where $T > 0$ is arbitrary but fixed. Finally, we specify assumptions that guarantee the existence of an optimal (two-dimensional) order scheme, i.e., a scheme that maximizes (4.67). The proofs of the following results are much like the proofs of the corresponding results in the previous section. For the sake of completeness and to emphasize particular differences, we (re)write the proofs at full length.

Proposition 4.3.6 *Assume all conditions of Proposition 4.3.1 hold true. Let $x_0 > 0$ be arbitrary but fixed, and let the minimum cycle length $\tau_0(x_0)$ exist, cf.* (4.58)*.*

Whenever there is a finite value $\bar{\tau}_\varnothing := \bar{\tau}_\varnothing(x_0)$, $\bar{\tau}_\varnothing \geqslant \tau_0(x_0)$, *such that* $\Pi_\varnothing(\bar{\tau}_\varnothing, x_0) \geqslant \Pi_\varnothing(\tau, x_0)$ *for all* $\tau > \bar{\tau}_\varnothing$, *then a maximizing cycle length* $\tau^\star_\varnothing(x_0)$, $\tau_0(x_0) \leqslant \tau^\star_\varnothing(x_0) \leqslant \bar{\tau}_\varnothing(x_0)$, *exists. Moreover,* $\tau^\star_\varnothing(x_0)$ *is a solution of the equation*

$$\Pi_\varnothing(\tau, x_0) = \dot{V}(\tau, x_0) - \dot{L}(\tau, x_0). \tag{4.68}$$

Proof. The two functions $1/\tau$ and $\Pi^\star_1(\tau, x_0)$ are differentiable in τ, cf. Proposition 4.3.1. Thus, the product $\Pi_\varnothing(\tau, x_0) = 1/\tau\,[\Pi^\star_1(\tau, x_0) - k]$ is differentiable (and continuous) too, $\tau > 0$. Moreover, $\Pi_\varnothing(\tau, x_0)$ is negative on $(0, \tau_0(x_0))$; recall, $\tau_0(x_0)$ is the smallest value such that $\Pi^\star_1(\tau_0(x_0), x_0) - k$ is nonnegative. Furthermore, by assumption, $\Pi_\varnothing(\tau, x_0)$ is bounded from above by the value $\Pi_\varnothing(\bar{\tau}_\varnothing, x_0)$ on $(\bar{\tau}_\varnothing, +\infty)$. Hence, the continuous function $\Pi_\varnothing(\tau, x_0)$ attains its global maximum in the interior of a compact interval which contains the interval $[\tau_0(x_0), \bar{\tau}_\varnothing(x_0)]$. Since $\Pi_\varnothing(\tau, x_0)$ is differentiable in τ, the first order condition

$$\frac{\partial \Pi_\varnothing}{\partial \tau}(\tau, x_0) = \frac{\dot{\Pi}^\star_1(\tau, x)\tau - (\Pi^\star_1(\tau, x) - k)}{\tau^2} = 0$$

is satisfied for $\tau^\star_\varnothing(x_0)$. Rearranging the first order condition yields equation (4.68). ♦

Remark 4.3.4 *We do not require the optimal value* $\Pi_\varnothing\left(\tau^\star_\varnothing(x_0), x_0\right)$ *to be strictly positive. In case the optimal value is zero, the monopolist is indifferent between entering the market and running no business. To make sure that the optimal value is strictly positive, it is sufficient to find some point where the function* $\Pi_\varnothing(\tau, x_0)$ *is positive. Note, the value of* $\Pi_\varnothing$ *at the end point* $\bar{\tau}_\varnothing$ *is allowed to be positive or negative. To find such a point* $\bar{\tau}_\varnothing$, *it is sufficient to find such a point for the function* $\Pi^\star_1(\tau, x_0)$.

Similar to the N-cycle case, the existence of an optimal (x_0-dependent) cycle length according to Proposition 4.3.6 does neither guarantee uniqueness of $\tau^\star_\varnothing(x_0)$ nor uniqueness of the solution of equation (4.68); additional information about the function $\Pi_\varnothing$ is needed. Nevertheless, equation (4.68) offers a nice economic interpretation: at an optimal point $\tau^\star_\varnothing(x_0)$ the average profit equals the marginal (one-cycle) profit with respect to time, i.e., at an optimal cycle length the marginal benefit of extending the length of one cycle balances the profit per time unit. This relation has the same interpretation as equation (4.34), $\pi_\varnothing(\tau) = \nu^*(\tau)$, in Theorem 4.2.3; $\nu^*(\tau)$ is the integrand of π^*_1 evaluated at the cycle length τ.

When the cycle length T is given, and we are maximizing $\Pi_\varnothing(T, x)$ with respect to x, we can apply the same reasoning as in the proof of Proposition 4.3.6: since the factor $1/T$ does not depend on x, the average profit function $\Pi_\varnothing$ is maximized with respect to

x whenever the one-cycle profit $\Pi_1^\star(T, x)$ is maximized with respect to x. For the sake of completeness, we restate proper existence conditions in terms of x, and we do not simply refer to Proposition 4.3.6 for the analogous conditions related to the variable τ. We omit the proof of the following result; it runs along the same lines as the proof of Proposition 4.3.6.

Proposition 4.3.7 *Assume all conditions of Proposition 4.3.1 hold true. Let $T > 0$ be arbitrary but fixed, and let the minimum capacity $\chi_0(T)$ exist, cf.* (4.59).

Whenever there is a finite value $\bar{\chi} := \bar{\chi}(T)$, $\bar{\chi} \geqslant \chi_0(T)$, such that $\Pi_1^\star(T, \bar{\chi}) \geqslant \Pi_1^\star(T, x)$ for all $x > \bar{\chi}(T)$, then a maximizing capacity $x_\varnothing^\star(T)$, $\chi_0(T) \leqslant x_\varnothing^\star(T) \leqslant \bar{\chi}(T)$, exists. Moreover, $x_\varnothing^\star(T)$ is a solution of the equation

$$V'(T, x) = c_0 + L'(T, x). \tag{4.69}$$

Remark 4.3.5 *Except for the difference in meaning and the notation $x_\varnothing^\star(T)$ instead of $x^\star(T)$, the formulation of Proposition 4.3.7 is identical to the one of Proposition 4.3.5. If the parameter values are identical, then $x_\varnothing^\star(T) = x^\star(T)$. The statements of Remark 4.3.2 also remain valid when the objective is the profit per time unit. To verify the existence of a maximizing capacity, one can restrict the analysis to the analysis of the one-cycle profit function $\Pi_1^\star(T, x)$.*

Also, when the profit per time unit is the value to be maximized, the optimal inventory capacity does not depend on the (fixed) setup cost k. It only depends on the running cost and the purchasing cost (and market parameters). As before, additional information about the function $\Pi_\varnothing$ can facilitate the analysis.

Both propositions, 4.3.6 and 4.3.7, deal with one-dimensional optimization problems. The following theorem specifies assumptions that guarantee the existence of an optimal order scheme $\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$. This Theorem and its proof are very much alike Theorem 4.3.1 associated with the maximization of the present value of N cycles and its proof.

Theorem 4.3.2 *Assume all conditions of Proposition 4.3.1 hold true. There exists an optimal pair $\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$ of problem* (4.46), *if any of the following conditions,* (*i*), (*ii*), *or* (*iii*), *are satisfied:*

(*i*) (*a*) *There is a strip $\mathbb{R}_+ \times [x_1, x_2]$, $0 < x_1 < x_2 < \infty$, and a point $(\sigma, y) \in \Lambda_k$, $x_1 < y < x_2$, such that for all $(\tau, x) \in \Lambda_k, x \leqslant x_1$ or $x \geqslant x_2 : \Pi_\varnothing(\tau, x) < \Pi_\varnothing(\sigma, y)$.*

(*b*) *There is a continuous selection function $\tau_\varnothing^\star(x)$, $x \in [x_1, x_2]$, of solutions of the one-dimensional problems considered in Proposition 4.3.6.*

(ii) (a) *There is a strip* $[T_1, T_2] \times \mathbb{R}_+$, $0 < T_1 < T_2 < \infty$, *and a point* $(\sigma, y) \in \Lambda_k$, $T_1 < \sigma < T_2$, *such that for all* $(\tau, x) \in \Lambda_k, \tau \leqslant T_1$ *or* $\tau \geqslant T_2 : \Pi_\varnothing(\tau, x) < \Pi_\varnothing(\sigma, y)$.

(b) *There is a continuous selection function* $x^\star_\varnothing(T)$, $T \in [T_1, T_2]$, *of solutions of the one-dimensional problems considered in Proposition 4.3.7.*

(iii) There exists a rectangle $[T_1, T_2] \times [x_1, x_2]$, $0 < T_1 < T_2 < \infty, 0 < x_1 < x_2 < \infty$, *and a point* $(\sigma, y) \in \Lambda_k$, $T_1 < \sigma < T_2, x_1 < y < x_2$, *such that* $\Pi_\varnothing(\tau, x) < \Pi_\varnothing(\sigma, y)$ *for all* $\tau \notin (T_1, T_2)$ *and* $x \notin (x_1, x_2)$.

Proof. (i) By assumption, there exists a feasible pair (σ, y) such that $\Pi_\varnothing(\tau, x) < \Pi_\varnothing(\sigma, y)$ for all feasible pairs (τ, x) where $x \notin (x_1, x_2)$. Hence, if a global maximum of the function $\Pi_\varnothing(\tau, x)$ exists, this global maximum lies in the interior of the strip $\mathbb{R}_+ \times [x_1, x_2]$; recall that $\Pi^\star_1(0, x) = -c_0 x < 0$ for all $x > 0$, cf. (4.49). Moreover, the function $\Pi_\varnothing(\tau, x)$ is continuous in τ and in x (separately). Since, by assumption, a continuous selection function $\tau^\star_\varnothing(x), x \in [x_1, x_2]$ exists, the function $\Pi_\varnothing\left(\tau^\star_\varnothing(x), x\right)$ is continuous on the interval $[x_1, x_2]$ and will attain its global maximum at some point $x^\star_\varnothing$ in the interior of the interval $[x_1, x_2]$. Hence, an optimal order scheme $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ exists, where $\tau^\star_\varnothing := \tau^\star_\varnothing(x^\star_\varnothing)$.

(ii) The proof runs along the same lines as the proof of (i); simply change the point of view from the capacity dimension to the time dimension.

(iii) By assumption, there exists a feasible pair (σ, y) such that $\Pi_\varnothing(\tau, x) < \Pi_\varnothing(\sigma, y)$ for all feasible pairs (τ, x) outside or on the boundary of the rectangle $[T_1, T_2] \times [x_1, x_2]$. Hence, if a global maximum of the function $\Pi_\varnothing(\tau, x)$ exists, this global maximum lies in the interior of this rectangle. Since the function $\Pi_\varnothing(\tau, x)$ is continuous in τ and x (separately), it will attain its global maximum on $(T_1, T_2) \times (x_1, x_2)$. ♦

Remark 4.3.6 *Conditions* (i) *and* (ii) *in Theorem 4.3.1 are refined versions of* (iii). *We shall exploit the refined conditions when analyzing an illustrative example of problem* (4.48)*, see Section 4.3.3.*

Similar to the optimal order scheme $(\tau^\star_N, x^\star_N)$ that maximizes the present value of N cycles, the average-optimal order scheme $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ satisfies the optimality conditions of both one-dimensional subproblems.

Corollary 4.3.2 *Assume any of the conditions (i), (ii), or (iii) of Theorem 4.3.2 to be satisfied. Then, the optimal pair* $\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$ *solves the system of equations*

$$\Pi_{\varnothing}(\tau, x_0) = \dot{V}(\tau, x_0) - \dot{L}(\tau, x_0), \tag{4.70}$$

$$V'(\tau, x) = c_0 + L'(\tau, x). \tag{4.71}$$

The examples in the following section will illustrate the propositions of the one-dimensional subproblems and Theorem 4.3.2. Moreover, we analyze a very special case: the setting for the *endogenized Harris-Wilson* formula.

4.3.3 Illustrations and Examples

In Subsection 4.3.1, we formulate conditions that guarantee the existence of an optimal order scheme when the objective is to maximize the present value of N (identical) cycles and revenue maximizing price and advertising controls are applied. In Subsection 4.3.2, we analyze the case when the objective is to maximize the profit per time unit and formulate existence conditions. The following examples illustrate the meaning of such existence conditions, and we compare the results of both objectives with each other. In addition, we discuss a very special case of the time average problem for which we find an explicit formula of the optimal order scheme, the *endogenized Harris-Wilson* formula.

The following examples focus on the special case where the system function ψ equals one, i.e., the dynamic of the sales process does not depend on the inventory level itself. If $\psi \equiv 1$, the results of Section 4.2 can be applied and we can identify the optimal cycle length and inventory capacity when the profit maximizing pricing and advertising controls, see Chapter 2, are used. In the following, we apply the results of Section 4.3 which are based on the revenue maximizing pricing and advertising controls, see Chapter 3, and compare some of these results with those based on the results of Section 4.2. We like to stress once again that the approach which considers the (suboptimal) revenue maximizing policies is applicable in the general case of feedback functions depending on the current inventory level, i.e., $\psi(x)$ is not constant.

Example 4.3.1 *Assume all parameter values to be constant in time, i.e., let* $a > \delta > 0$, $\varepsilon > 1$, $\ell(t) \equiv \ell \geqslant 0$, *and* $\mu(t) \equiv \mu > 0$ *for all* $t \geqslant 0$; $\eta(t) \equiv \eta = \frac{\varepsilon - \Delta}{\Delta}\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon}\frac{\Delta}{\varepsilon-1}\mu\right]^{\frac{1}{1-\Delta}}$, *cf. Definition 3.2.1, and* $\gamma = \frac{\varepsilon-\Delta}{1-\Delta}$. *Let* $r \geqslant 0$ *and* $\psi(x) \equiv 1$. *Then,*

$$A^{(0)}(t, \tau) = \frac{\eta}{\gamma r} e^{-\gamma r \tau}\left(e^{\gamma r(\tau - t)} - 1\right) \underset{r \to 0}{\longrightarrow} \eta(\tau - t),$$

$$B(x) = \frac{\gamma}{\gamma - 1} x,$$

and

$$V(\tau, x) = A^{(0)}(0, \tau)^{\frac{1}{\gamma}} B(x)^{\frac{\gamma-1}{\gamma}} = c_V \left(\frac{1 - e^{-\gamma r \tau}}{\gamma r} \right)^{\frac{1}{\gamma}} x^{\frac{\gamma-1}{\gamma}} \underset{r \to 0}{\longrightarrow} c_V \tau^{\frac{1}{\gamma}} x^{\frac{\gamma-1}{\gamma}},$$

where $c_V = \frac{\gamma}{\gamma-1}^{\frac{\gamma-1}{\gamma}} \eta^{\frac{1}{\gamma}} > 0$.

If $r \equiv 0$, *the model ensures that the inventory depletes at a constant rate* $\frac{x}{\tau}$; *if* $r > 0$, *the revenue maximizing trajectory is a convex function in* t:

$$x^{\star}(t, \tau, x) = \frac{e^{\gamma r (\tau - t)} - 1}{e^{\gamma r \tau} - 1} x \underset{r \to 0}{\longrightarrow} \left(1 - \frac{t}{\tau} \right) x,$$

The associated total running costs are given by

$$L(\tau, x) = \begin{cases} \dfrac{\ell}{(\gamma + 1)\, r} \left(1 - \gamma \dfrac{1 - e^{-r\tau}}{e^{\gamma r \tau} - 1} \right) x, & \text{if } r > 0, \\ \dfrac{\ell}{2} \tau x, & \text{if } r = 0. \end{cases} \tag{4.72}$$

The present value of N *cycles is given by,* $r > 0$,

$$\Pi_N(\tau, x) = \frac{1 - e^{-NrT}}{1 - e^{-rT}} \left[c_V \left(\frac{1 - e^{-\gamma r \tau}}{\gamma r} \right)^{\frac{1}{\gamma}} x^{\frac{\gamma-1}{\gamma}} - c_0 x - \frac{\ell x}{(1 + \gamma)\, r} \left(1 - \gamma \frac{1 - e^{-r\tau}}{e^{\gamma r \tau} - 1} \right) - k \right];$$

if $r \equiv 0$, *the average profit per time unit is given by*

$$\Pi_{\varnothing}(\tau, x) = \frac{c_V \tau^{\frac{1}{\gamma}} x^{\frac{\gamma-1}{\gamma}} - c_0 x - \frac{\ell}{2} \tau x - k}{\tau}.$$

Since $\gamma > 1$, the revenue function $V(\tau, x)$ is concave in τ and in x. Evidently, the expressions of interest are easier to analyze when there is no discounting: if $r \equiv 0$, the (*production*) function V is a homogeneous *Cobb-Douglas* function in the (*production*) factors τ and x; the function $L(\tau, x)$, the running costs, is a linear function in τ and x. We will examine problem (4.46), the maximization of $\Pi_{\varnothing}(\tau, x)$, in detail; problem (4.48), the maximization of the N-cycle profit runs along similar lines. Although we are interested in an optimal two-dimensional order scheme $\left(\tau^{\star}_{\varnothing}, x^{\star}_{\varnothing}\right)$, we will consider each one-dimensional subproblem separately. We shall first consider the case when x_0 is exogenously given and solve for $\tau^{\star}_{\varnothing}(x_0)$. Afterwards, we shall keep T fixed and solve for $x^{\star}_{\varnothing}(T)$ before identifying the best order scheme $\left(\tau^{\star}_{\varnothing}, x^{\star}_{\varnothing}\right)$ value.

Assume the capacity x_0 to be arbitrary but fixed. According to Proposition 4.3.6, we require the market-entry condition to be satisfied, i.e., there is a value $\tau_0(x_0)$ such that $\Pi_1^\star(\tau_0(x_0), x_0) \geqslant k$. More precisely, $\tau_0(x_0)$ is the smallest value that satisfies this inequality. Since all terms of the functional expression $\Pi_1^\star(\tau, x_0)$ are continuous in τ, $\tau_0(x_0)$ is the smallest (positive) solution of

$$c_V x_0^{\frac{\gamma-1}{\gamma}} \tau^{\frac{1}{\gamma}} = k + c_0 x_0 + \frac{\ell}{2} x_0 \tau. \tag{4.73}$$

The left-hand side of equation (4.73) is strictly increasing and concave in τ; at $\tau = 0$ it takes the value zero. The right-hand side of equation (4.73) is a linear expression in τ that takes the value $k + c_0 x_0 > 0$ at $\tau = 0$. Hence, there are three possible cases: equation (4.73) has (*i*) no solution, (*ii*) one solution, or (*iii*) two solutions. The left panel of Figure 4.6 illustrates the three cases. The blue curve is the graph of the left-hand side of equation (4.73), the revenue function $V(\tau, x_0)$, as a function of τ. The three straight lines represent the right-hand side of equation (4.73) for three distinct parameter choices of the setup cost k. Likewise, one can analyze the three cases by changing the left-hand side of (4.73), e.g., by choosing different values for the arrival intensities, or adjusting another parameter value accordingly; however, the characterization in terms of the three cases (*i*), (*ii*), and (*iii*) remains unaffected.

The (cost) lines are parallel lines with slope $\frac{\ell}{2} x_0$ and intersect the ordinate at (different values) $k + c_0 x_0$. If k is (*too*) large - this is case (*i*) and is indicated by the yellow line - there is no cycle length value such that the costs are covered. The market situation is not profitable; the high fixed cost prevents the retailer from entering the market.[38] In case (*ii*), the red straight line, there is exactly one τ value that guarantees nonnegative profits. However, in this particular case the retailer can not attain a positive profit, and she is indifferent between entering the market or not, cf. Remark 4.3.4. Since for this particular capacity value x_0 there is only one feasible order scheme, the set $\Lambda_k = \{(\tau_0(x_0), x_0)\}$ is a singleton, and the minimum cycle length is also the optimal cycle length, i.e., $\tau_\varnothing^\star(x_0) = \tau_0(x_0)$. Case (*iii*), illustrated by the green line, is the *standard* case: the first point where the blue curve and the green line intersect is the minimum cycle length $\tau_0(x_0)$, the second point is a maximum cycle length $\bar{\tau}(x_0)$: for values larger than $\bar{\tau}(x_0)$, the profit of one cycle and thus the profit per time unit is negative. Hence, the set of feasible order schemes for this one-dimensional subproblem is given by $\Lambda_k = \{(\tau, x_0) | \tau_0(x_0) \leqslant \tau \leqslant \bar{\tau}(x_0)\}$. The market situation is so that the monopolist can choose from a range of cycle length values. Among those choices she has to identify the optimal cycle length.

[38] Since the capacity x_0 is assumed to be fixed, the unit costs $c_0 x_0$ are also fixed.

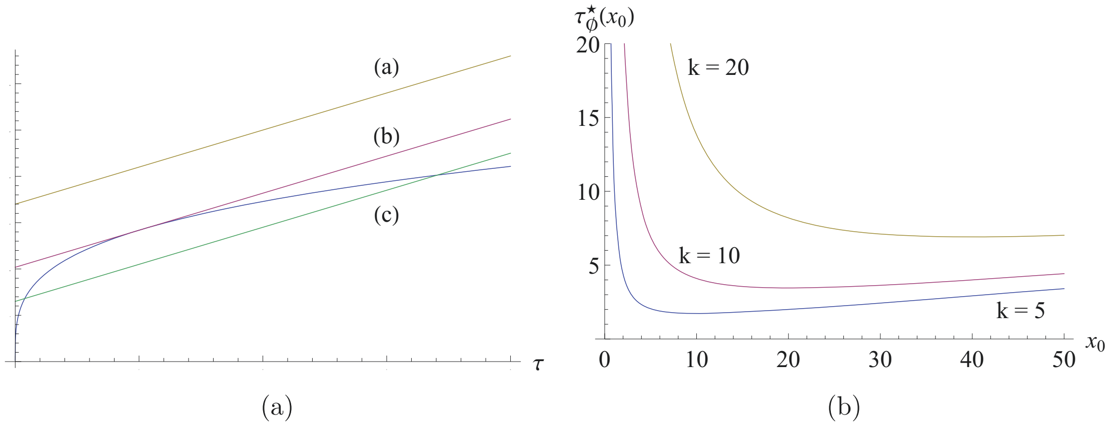


Figure 4.6: Panel (a): illustration of equation (4.73) when the capacity x_0 is exogenously given. The blue curve shows the maximized revenue and the yellow, red, and green line represent different cost functions in case $(i), (ii)$, and (iii), see text. Panel (b): the average-optimal cycle length as a function of x_0 according to (4.74) for different values of k, where $\varepsilon = 2, \Delta = 0.5, \mu = 25, c_0 = 1, \ell = 0.1, r = 0$.

If we consider the one-dimensional subproblem in x when the cycle length T is exogenously given and fixed, the analysis of equation (4.73) is essentially the same as in the case when x is unknown. Replacing τ by (the fixed number) T and the fixed value x_0 by x, the left-hand side of (4.73) is still a strictly increasing and concave function (now in x) starting at the origin; the right-hand side is again a linear expression (now in x). Although the numerical values change due to different values of the exponents and the factors, the basic properties of equation (4.73) are preserved.[39] In particular, we will also have to deal with all three possible cases (i), (ii), and (iii), see above.

In the following, we assume the set of feasible order schemes is non-empty. Then, there exists an optimal solution for each one-dimensional subproblem. We evaluate the first order condition (4.68) and solve for τ; elementary algebra shows that the average-optimal cycle length for a predefined capacity x_0 is given by

$$\tau^\star_{\varnothing}(x_0) = \frac{\gamma}{\gamma - 1}\frac{x_0}{\eta}\left(c_0 + \frac{k}{x_0}\right)^{\gamma}. \tag{4.74}$$

The optimal cycle length $\tau^\star_{\varnothing}(x_0)$ increases in the unit cost c_0 and the fixed cost k but does not depend on the inventory cost parameter ℓ. For *large* capacity values, $\tau^\star_{\varnothing}(x_0)$ increases in x_0: the more items will be sold the more time is allocated. To see this

[39]Note, the setup cost k now solely determines the intercept of the right-hand side of (4.73) with the ordinate. The unit cost c_0 enters the slope expression together with the term $\ell T/2$.

fact, notice that the term k/x_0 within the parentheses of the right-hand side of equation (4.74) becomes small, if x_0 is large, and the linear factor x_0 becomes the dominating one. However, if x_0 takes *small* values the average-optimal cycle length will decrease in x_0; recall, $\gamma > 1$, and the term $(c_0 + k/x_0)^\gamma$ can get large. Elementary calculations show that the expression $\tau^\star_\varnothing(x_0)$ attains its minimum value at

$$x_0 = \frac{\gamma - 1}{c_0} k, \tag{4.75}$$

and strictly increases to the left and to the right of x_0. Hence, if x_0 is fixed, a lower bound for the average-optimal cycle length is given by

$$\tau^\star_\varnothing\left(\frac{\gamma-1}{c_0}k\right) = c_0^{\gamma-1} k \frac{\gamma}{\eta}\left(\frac{\gamma}{\gamma-1}\right)^\gamma .$$

If not otherwise specified, we assume the following parameter values for the numerical analysis of the average profit per time unit.

Example 4.3.1 (continued) *Let* $\mu = 25, a = 1, \delta = 0.5, \varepsilon = 2, c_0 = 1, \ell = 0.1,$ *and* $r = 0$. *Then,* $\Delta = 0.5$, $\gamma = 3$, $\eta = \frac{1875}{64}(\approx 29)$, $c_V = 3\left(\frac{5}{4}\right)^{\frac{4}{3}} (\approx 4)$, $A^{(0)}(0,\tau) = \frac{1875}{64}\tau$, *and* $B(x) = \frac{3}{2}x$ *such that* $V(\tau, x) = 3\left(\frac{5}{4}\right)^{\frac{4}{3}} \tau^{\frac{1}{3}} x^{\frac{2}{3}}$ *and* $\Pi_\varnothing(\tau, x) = \frac{3\left(\frac{5}{4}\right)^{\frac{4}{3}}\tau^{\frac{1}{3}}x^{\frac{2}{3}} - x - 0.2\tau x - k}{\tau}$.

For the parameter setting of Example 4.3.1 a feasible cycle length τ must satisfy the inequality

$$3\left(\frac{5}{4}\right)^{\frac{4}{3}} x_0^{\frac{2}{3}} \tau^{\frac{1}{3}} > k + x_0 + \frac{1}{20} x_0 \tau,$$

cf. (4.73). For instance, if $k = 10$ and $x_0 = 20$, the minimum cycle length is $\tau_0(20) \approx 1.15$. The average-optimal cycle length in this case is $\tau^\star_\varnothing(20) = \frac{432}{125} \approx 3.46$; note, formula (4.74) becomes

$$\tau^\star_\varnothing(x_0) = \frac{32}{625} x_0 \left(1 + \frac{k}{x_0}\right)^3,$$

with the numerical values of Example 4.3.1 in place. Panel (b) of Figure 4.6 shows the average-optimal cycle length depending on x_0 for three different values of k. The graphs show a convex shape; each line attains its minimum at the corresponding value where $x_0 = \frac{\gamma-1}{c_0}k = 2k$, cf. (4.75).

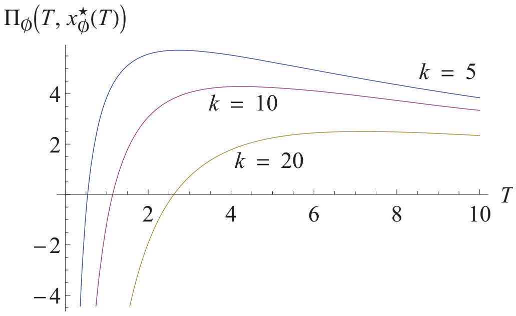
If the sales horizon T is fixed, the problem is to choose the average-optimal capacity. According to Proposition 4.3.7, the (optimal) value $x^\star_\varnothing(T)$ is a solution of (4.69), $V'(T, x) = c_0 + L'(T, x)$. In the time-homogeneous setting of Example 4.3.1, the average-optimal capacity depending on T is given by

$$x^\star_\varnothing(T) = \frac{\gamma - 1}{\gamma}\,\eta\,\frac{T}{\left(c_0 + \frac{\ell}{2}T\right)^\gamma}. \tag{4.76}$$

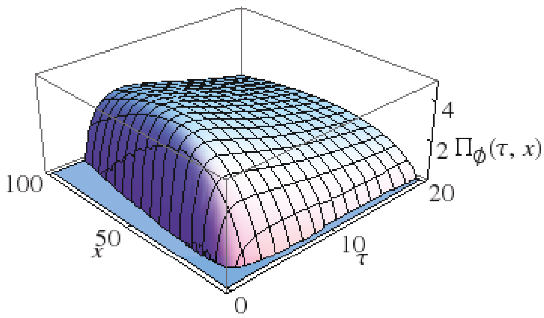
In contrast to the optimal cycle length $\tau^\star_\varnothing$ in (4.74), the optimal capacity $x^\star_\varnothing$ decreases in c_0 and does not depend on the fixed cost k. This latter fact is not surprising and has been discussed before, cf. the paragraphs after Remark 4.3.2 and Remark 4.3.5. Instead, the inventory cost parameter ℓ influences the optimal choice of the capacity: the larger the value of ℓ, the smaller is the optimal capacity value. The way $x^\star_\varnothing$ depends on the exogenously given factor - now the sales horizon - also changes compared to (4.74): $x^\star_\varnothing(T)$ is a unimodal function of T; it increases for *small* values of T and decreases if T becomes *large*. The value of $x^\star_\varnothing(T)$ reaches its maximum - the *upper* bound on the optimal capacity - at

$$T = \frac{2c_0}{(\gamma - 1)\ell}.$$

Since $x^\star_\varnothing(T)$ does not depend on k, for the parameter setting of Example 4.3.1 the



(a)



(b)

Figure 4.7: Panel (a) shows the average-optimal profits depending on the cycle length T for different values of k when $x^\star_\varnothing$ is chosen according to (4.76), where $\varepsilon = 2, \Delta = 0.5$, $\mu = 25, c_0 = 1, \ell = 0.1, r = 0$. Panel (b): the function $\Pi_\varnothing(\tau, x)$ for the parameter setting of Example 4.3.1 and $k = 10$.

maximum value for the (optimal) capacity (4.76) is attained at $T = 10$ and is (approximately) $x^\star_\varnothing(10) \approx 57.87$. For different k values, panel (a) of Figure 4.7 shows the average-optimal profit $\Pi_\varnothing\left(T, x^\star_\varnothing(T)\right)$. Although we are only interested in order schemes that guarantee a nonnegative (average) profit, we show the plots including negative values of $\Pi_\varnothing\left(T, x^\star_\varnothing(T)\right)$ to emphasize once again the importance of the minimum capacity value $\chi_0(T)$, cf. (4.73). On the domain $(0, 10)$ the average profit function takes positive and negative values. As the graphs illustrate, the average profit per time unit sharply increases in a neighborhood of the minimum cycle length $\chi(T)$. Hence, simply exploiting (4.76) can lead to substantial losses if $x^\star_\varnothing(T) < \chi(T)$. Whenever this strict inequality holds, it is optimal to run no business at all; the analogous analysis naturally applies to the case where the capacity x_0 is fixed and the quantity to choose is the cycle length, see above.

Finally, we consider the problem when the monopolist may choose both values of interest, the cycle length and the capacity. According to Corollary 4.3.2, the optimal order scheme must satisfy the system of equations (4.70) and (4.71). In the time-homogeneous setting of Example 4.3.1, this system is equivalent to (4.74) and (4.76) when x replaces x_0 and τ replaces T. It is easy to verify that an optimal order scheme $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ satisfies the equation

$$\tau^\star_\varnothing x^\star_\varnothing = \frac{2k}{\ell}. \tag{4.77}$$

The same condition is satisfied for the very basic EOQ model, the classic Harris-Wilson model; in our terms expressed as $\tau_{HW} x_{HW} = 2k/\ell$, see Section 4.1. The presence of pricing and advertising considerations has no influence on the product of optimal cycle length and optimal capacity in the time-homogeneous setting! Moreover, this product only depends on the setup cost k and the inventory cost parameter ℓ but neither on the unit cost, on the arrival intensity, nor on the price and advertising elasticities. Recall, $L(\tau, x) = \frac{\ell}{2}\tau x$. Hence, equation (4.77) can be rewritten as $L\left(\tau^\star_\varnothing, x^\star_\varnothing\right) = k$, i.e., a necessary optimality condition requires the order scheme to be such that the total inventory costs equal the setup cost.

To determine the optimal values $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$, one can either solve the system of equations (4.70) and (4.71) (and verify that a candidate solution actually guarantees a nonnegative profit) or make use of the solution of the one-dimensional subproblems $\tau^\star_\varnothing(x_0)$, $x^\star_\varnothing(T)$ resp., and maximize $\Pi_\varnothing\left(\tau^\star(x), x\right)$ with respect to x, maximize $\Pi_\varnothing\left(\tau, x^\star_\varnothing(\tau)\right)$ with respect to τ resp. Visual inspection of the graphs of $\Pi_\varnothing\left(T, x^\star_\varnothing(T)\right)$ in panel (a) of Figure 4.7 suggests numerical values of the optimal cycle length. For example, if $k = 20$, one expects $\tau^\star_\varnothing$ to take approximately the value 7. In general - and independent of the approach being taken - the optimal order scheme can only be numerically determined. For different setup cost values k Table 4.3 shows the optimal order scheme (column two and three) and the associated average profit per time unit (column four) for the parameter setting of Example 4.3.1. Formulas (4.74) and (4.77) suggest that the optimal cycle length and the optimal capacity value increase in k; naturally, the (optimal) average profit decreases in k (and in any other cost parameter). Columns five to seven of Table 4.3 show the corresponding optimal values when the production and inventory costs are

k	$\tau_\varnothing^\star$	$x_\varnothing^\star$	$\Pi_\varnothing\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$	$\tau_\varnothing$	$x_\varnothing$	$\pi_\varnothing(\tau_\varnothing)$	percentage gain
5	2.74	36.44	5.73	2.92	39.19	5.85	2.04
10	4.28	46.72	4.29	4.71	52.50	4.52	5.27
20	7.16	55.84	2.50	8.27	68.39	2.93	16.97

Table 4.3: For Example 4.3.1 and different values of k: optimal order schemes $\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$ based on the revenue maximizing policies within a cycle and associated average profits per time unit $\Pi_\varnothing\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$, and $(\tau_\varnothing, x_\varnothing)$ based on the profit maximizing policies within a cycle and associated average profit per time unit $\pi_\varnothing(\tau_\varnothing)$. The last column shows the percentage gain per time unit by $\pi_\varnothing(\tau_\varnothing)$ over $\Pi_\varnothing\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$.

included in the (optimal) pricing-advertising control. The last column displays the gain of the profit maximizing strategy (Chapter 2 and Section 4.2) over the revenue maximizing strategy (Chapter 3 and Section 4.3). This gain is expressed as the percentage of $\left(\pi_\varnothing(\tau_\varnothing) - \Pi_\varnothing\left(\tau_\varnothing^\star, x_\varnothing^\star\right)\right)/\Pi_\varnothing\left(\tau_\varnothing^\star, x_\varnothing^\star\right)$. For example, if $k = 10$ (second row), then the benefit from integrating the production and inventory cost c_0 and ℓ into the pricing-advertising strategy leads to a net profit increase of approximately five percent per time unit. Recall, for the parameter setting of Example 4.3.1 - the time-homogeneous case without discounting - the revenue maximizing strategy, see Theorem 3.2.1, is to set a constant price and a constant advertising rate such that the inventory depletes at a linear rate x/τ. In particular, if $k = 10$, then $p^\star(t) \equiv 2.43$ and $w^\star(t) \equiv 6.63$.[40] Since the time-to-go potential $A^{(0)}$ depends on the value of the (optimal) cycle length and the diffusion potential B on the (optimal) capacity value, cf. Definition 3.2.1 in Chapter 3, both values - the optimal price and the optimal advertising level - depend on the optimal order scheme and thus implicitly on the setup cost k. In contrast to the revenue maximizing strategy, the profit maximizing strategy, see Theorem 2.2.1, is only influenced by the order scheme by the length of time it is applied. For our (numerical) example, the price that maximizes the net profit rate is given by $p^*(t) = \varepsilon/(\varepsilon-1)c(t) = 2 + 0.2t$ and the corresponding advertising rate by $w^*(t) = c_w\left(\mu/(c(t)^{\varepsilon-1}\right)^{1/(a-\delta)} \approx 9.76/(1+0.1t)^2$, cf. Theorem 2.2.1. The price function p^* monotonically increases (linearly) over time and the advertising rate monotonically decreases over time.

The preceding examples illustrate that, even for rather simple model settings (time-homogeneous parameter values and state-independent demand), determining feasible order schemes and an optimal order scheme is not easy. Notably, the optimal order

[40] Recall, that for our numerical example $A^{(0)}(0,\tau) = \frac{1875}{64}\tau$ and $B(x) = \frac{3}{2}x$. According to Theorem 3.2.1 the revenue maximizing price and advertising rate for an order scheme (τ, x) are given by $p^\star := p^\star(t;\tau,x) = \frac{\varepsilon}{\varepsilon-1}\left(\frac{A^{(0)}(0,\tau)}{B(x)}\right)^{\frac{1}{\gamma}}$ and $w^\star := w^\star(t,\tau,x) = \frac{625}{64}\left(\frac{32}{625}\frac{x}{\tau}\right)^{\frac{2}{3}}$. For the order scheme $(\tau_\varnothing^\star, x_\varnothing^\star) \approx (4.28, 46.72)$ the values $p^\star \equiv 2.43$ and $w^\star \equiv 6.63$ follow.

scheme can only be numerically determined, a fact, which often implies further challenges. However, we find a particular - and only one - parameter setting where the optimal order scheme can be described in terms of an elementary expression, the *endogenized Harris-Wilson* formula.

Proposition 4.3.8 *Let* $\delta = r \equiv 0, \psi(x) \equiv 1, \varepsilon \equiv 2, c_0 > 0, k > 0, \ell(t) \equiv \ell > 0,$ *and* $\mu(t) \equiv \mu > 2k\ell$. *Then, the optimal order scheme* $(\tau^\star_{HW}, x^\star_{HW})$ *that maximizes the average profit per time unit* (4.3.2) *- the endogenized Harris-Wilson formula - is given by*

$$\tau^\star_{HW} = \sqrt{\frac{2k}{\ell}} \frac{2c_0}{\sqrt{\mu} - \sqrt{2k\ell}} \tag{4.78}$$

and

$$x^\star_{HW} = \sqrt{\frac{2k}{\ell}} \frac{\sqrt{\mu} - \sqrt{2k\ell}}{2c_0}. \tag{4.79}$$

The optimal average profit per time unit $\Pi^\star_{HW}$ *is given by*

$$\Pi^\star_{HW} := \Pi_\varnothing\left(\tau^\star_{HW}, x^\star_{HW}\right) = \frac{\left(\sqrt{\mu} - \sqrt{2k\ell}\right)^2}{4c_0}. \tag{4.80}$$

Proof. We make use of the formulas derived in Example 4.3.1; note, if $\delta \equiv 0$, then $\gamma = \varepsilon = 2$, $\eta = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1} \mu = \mu/2$, $A^{(0)}(0,\tau) = \frac{\mu}{2}\tau$ and $B(x) = 2x$. Thus, the revenue function V is given by

$$V(\tau, x) = \sqrt{\mu\tau x}.$$

Since $r \equiv 0$ and $\psi(x) \equiv 1$, the inventory depletes at a constant rate x/τ; hence, the total inventory cost is given by (4.72), $L(\tau, x) = \frac{\ell}{2}\tau x$, cf. Example 4.3.1. Thus, the objective function to be maximized with respect to τ and x is given by

$$\Pi_\varnothing(\tau, x) = \frac{V(\tau, x) - (c_0 x + L(\tau, x) + k)}{\tau} = \frac{\sqrt{\mu\tau x} - \left(c_0 x + \frac{\ell}{2}\tau x + k\right)}{\tau}.$$

The first order conditions (4.74) and (4.76) imply that the optimal order scheme $(\tau^\star_{HW}, x^\star_{HW})$ needs to satisfy the following system of equations:

$$\begin{aligned} \tau^\star_{HW} &= \frac{4}{\mu} x^\star_{HW} \left(c_0 + \frac{k}{x^\star_{HW}}\right)^2, \\ x^\star_{HW} &= \frac{\mu}{4} \frac{\tau^\star_{HW}}{\left(c_0 + \frac{\ell}{2}\tau^\star_{HW}\right)^2}. \end{aligned}$$

Formulas (4.78) and (4.79) follow by making use of the corresponding version of equation

(4.77), $\tau^{\star}_{HW} x^{\star}_{HW} = \frac{2k}{\ell}$, and basic algebra. The expression for the maximized average profit $\Pi_{\varnothing}\left(\tau^{\star}_{HW}, x^{\star}_{HW}\right)$ is an immediate consequence of both formulas. ♦

In contrast to the classical Harris-Wilson EOQ our Harris-Wilson formula *endogenizes* the demand since the demand rate is constant over time $(x^{\star}_{HW}/\tau^{\star}_{HW})$ but *not* exogenously given. And not only the demand rate is endogenized but also the optimal price $p^{\star}(t) = \frac{\mu}{2}\frac{\tau}{x}$.[41] Due to this fact, we call it the *endogenized Harris-Wilson* formula. Furthermore, the order scheme according to the endogenized Harris-Wilson formula maximizes the average profit per time unit whereas the classical EOQ considers the minimization of the average costs per time unit.

At first glance, the results of Proposition 4.3.8 seem to be rather limited and very special due to the restrictive parameter setting: a pure pricing model ($\delta \equiv 0$) and constant cost parameter values as well as a constant arrival intensity. However, the endogenized Harris-Wilson formula reveals the structural properties of an optimal order scheme. To this end, we first collect some facts about the endogenized Harris-Wilson formula.

Remark 4.3.7 *The optimal order scheme according to the endogenized Harris-Wilson formula, equations* (4.78) *and* (4.79) *in Proposition 4.3.8, implies the following results:*

- *The optimal cycle length increases in the unit cost* c_0 *and decreases in the inventory parameter* ℓ.
- *The optimal order quantity decreases in* c_0 *and increases in* ℓ.
- *Both increase in the setup cost* k.
- *The optimal cycle length decreases in the arrival intensity* μ.
- *The optimal order quantity increases in* μ.

Moreover, due to the square root function in all the expressions the endogenized Harris-Wilson formula is robust with respect to misspecifications and uncertainties of all parameters. This feature is well known for the *classical* Harris-Wilson formula. Table 4.4 illustrates the robustness of the endogenized Harris-Wilson formula. The first row of Table 4.4 shows the revenue maximizing order scheme $\left(\tau^{\star}_{\varnothing}, x^{\star}_{\varnothing}\right)$ for a particular parameter setting: $\delta = 0, \varepsilon = 2, \mu = 10, r \equiv 0, c_0 = 1, k = 10, \ell = 0.1$. For this setting, the

[41] According to Theorem 3.2.1 the revenue maximizing price for an order scheme (τ, x) is given by $p^{\star}(t) = p^{\star}(t; \tau, x) = \frac{\varepsilon}{\varepsilon - 1}\left(\frac{A^{(0)}(0,\tau)}{B(x)}\right)^{\frac{1}{\gamma}}$. Since in the special case of Proposition 4.3.8 $\gamma = \varepsilon = 2$, $A^{(0)}(0,\tau) = \frac{\mu}{2}\tau$, and $B(x) = 2x$, $p^{\star}(t)\frac{\mu}{2}\frac{\tau}{x}$ follows.

			revenue maximizing		Harris-Wilson	
c_0	k	ℓ	$\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$	$\Pi_\varnothing\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$	$\Pi_\varnothing\left(\tau^\star_{HW}, x^\star_{HW}\right)$	%
1	5	0.1	$(9.25, 10.81)$	1.17	1.17	100%
0.8	5	0.1	$(7.40, 13.51)$	1.46	1.40	95.9%
1.2	5	0.1	$(11.10, 9.01)$	0.97	0.94	96.9%
1	2	0.1	$(5.00, 8.00)$	1.60	1.49	93.1%
1	8	0.1	$(13.33, 12.00)$	0.90	0.84	93.3%
1	5	0.05	$(11.52, 17.36)$	1.50	1.44	96.0%
1	5	0.2	$(8.09, 6.18)$	0.76	0.63	82.9%

Table 4.4: Optimal order schemes $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ applying the revenue maximizing price-advertising policy and associated average profits $\Pi_\varnothing\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ for the parameter setting $\delta \equiv 0$, $\varepsilon \equiv 2$, $\mu \equiv 10$, $r \equiv 0$, $\psi(x) \equiv 1$, and c_0, k, and ℓ as indicated in the first three columns. $\Pi_\varnothing\left(\tau^\star_{HW}, x^\star_{HW}\right)$ is the average profit when the order scheme $(9.25, 10.81)$ is applied in each case. The values in the last column are the percentage values of $\Pi_\varnothing\left(\tau^\star_{HW}, x^\star_{HW}\right)$ compared to $\Pi_\varnothing\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$.

optimal order scheme is identical to the order scheme specified by the endogenized Harris-Wilson formula, i.e., $\left(\tau^\star_\varnothing, x^\star_\varnothing\right) = \left(\tau^\star_{HW}, x^\star_{HW}\right)$. The subsequent rows of Table 4.4 show the optimal order scheme $\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ and the associated profit per time unit $\Pi_\varnothing\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$ for the parameter values displayed in the first three columns (a selection of values for c_0, k, and ℓ). The sixth column - indicated by the column header $\Pi_\varnothing\left(\tau^\star_{HW}, x^\star_{HW}\right)$ - displays the profit per time unit for the suboptimal Harris-Wilson policy specified in row one. In fact, this situation assumes that the decision maker is not aware of the modified parameter setting. The last column of Table 4.4 contains the percentage values of the profit corresponding to the suboptimal Harris-Wilson policy and the optimal profit $\Pi_\varnothing\left(\tau^\star_\varnothing, x^\star_\varnothing\right)$. For example, in the second row, if $c_0 = 0.8$ (instead of $c_0 = 1$), the optimal order scheme is $\left(\tau^\star_\varnothing, x^\star_\varnothing\right) = (7.40, 13.51)$ which yields a profit of 1.46 monetary units per time unit. Applying the suboptimal Harris-Wilson policy $\left(\tau^\star_{HW}, x^\star_{HW}\right) = (9.25, 10.81)$ leads to a profit of 1.40 monetary units per time unit[42] - more than 95 percent of the optimal value. Although the *true* unit cost has been overestimated by more than 20 percent (1 instead of 0.8) the retailer loses only roughly four percent of the profit per time unit. This *robustness* feature is not limited to changes in the cost parameters but can also be observed with regard to misspecifications of the demand parameters. The structure of Table 4.5 is similar to the structure of Table 4.4. The first row lists the identical parameter setting as the first row of Table 4.4. Rows two to seven of Table 4.5

[42]The numerical values are rounded to two decimal places.

specify parameter variations as indicated by the first three columns. The loss induced

			revenue maximizing		Harris-Wilson	
ε	μ	$\psi(x)$	$\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$	$\Pi_{\varnothing}\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$	$\Pi_{\varnothing}\left(\tau_{HW}^{\star}, x_{HW}^{\star}\right)$	%
2	10	1	$(9.25, 10.81)$	1.17	1.17	100%
1.1	10	1	$(16.45, 6.08)$	6.43	5.98	93.0%
3	10	1	$(11.46, 8.72)$	0.16	0.14	87.5%
2	5	1	$(16.18, 6.18)$	0.38	0.17	44.7%
2	20	1	$(5.76, 17.36)$	3.01	2.58	85.7%
2	10	$x^{-0.2}$	$(9.44, 6.45)$	0.93	0.82	88.1%
2	10	$x^{0.2}$	$(8.61, 22.24)$	1.95	1.66	85.1%

Table 4.5: Optimal order schemes $\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$ applying the revenue maximizing price-advertising policy and associated average profits $\Pi_{\varnothing}\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$ for the parameter setting $\delta \equiv 0, r \equiv 0, c_0 \equiv 1, k \equiv 5$, and $\ell \equiv 0.1$ and ε, μ, and $\psi(x)$ as indicated in the first three columns. $\Pi_{\varnothing}\left(\tau_{HW}, x_{HW}\right)$ is the average profit when the order scheme $(9.25, 10.81)$ is applied in each case. The values in the last column are the percentage values of $\Pi_{\varnothing}\left(\tau_{HW}^{\star}, x_{HW}^{\star}\right)$ compared to $\Pi_{\varnothing}\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$.

by applying the suboptimal Harris-Wilson policy instead of applying the optimal order scheme $\left(\tau_{\varnothing}^{\star}, x_{\varnothing}^{\star}\right)$ is larger than in the case of Table 4.4. However, the endogenized Harris-Wilson policy still provides a reasonable decision rule when the true parameter values are unknown (or when estimates of the parameters are biased). The numbers suggest using conservative estimates of the parameters in practical applications. For example, if the true arrival intensity equals 20 but the retailer assumes $\mu = 10$, then the suboptimal average profit still accounts for 85 percent of the optimal one, see row five of Table 4.5. However, if the true value is $\mu = 5$, applying the suboptimal order scheme leads to a profit setback of more than 50 percent.[43]

Eventually, the endogenized Harris-Wilson formula is more than just an explicit solution formula for a very special parameter setting: it provides useful insight into the dependence of the (optimal) order scheme on the model parameters and can be used as a robust decision rule.

[43]Note, since the average profit heavily depends on the μ value the absolute values differ substantially in both cases.

5 Summary and Conclusion

In this thesis, we consider the optimal coordination of dynamic pricing, dynamic advertising, and inventory decisions. In Chapter 2, we extend the dynamic pricing model of Rajan et al. (1992) by the opportunity to dynamically advertise and by (time-dependent) discounting effects. For a fixed sales horizon we derive an explicit expression for an optimal (dynamic) price-advertising control and an optimal capacity that maximize the net present profit value of a monopolist. The optimal price depends on a cost function accounting for purchasing and inventory costs as well as interest and deterioration effects, and can be thought of as a markup, determined by the price elasticity of demand, of the cost function. Typically, optimal prices increase over time. The behavior of the optimal advertising control over time depends, in particular, on the ratio of the customer arrival intensity and the cost function; typically, it is a decreasing function. The optimal capacity value can be computed once the optimal price-advertising control is known. We also consider optimal combinations of static and dynamic controls when either the price or the advertising rate is allowed to vary over time and the other control variable is constant. In general, the dynamic component of the optimal partially static control inherits the essential structural characteristics of the fully dynamic optimal policy. A key result of our analysis is the following one: if it is optimal to advertise at a rate greater than one, the opportunity of advertising benefits the monopolist *and* the customers. The monopolist is able to increase its profit, and more customers will purchase the product for the same (optimal) price as in the pure pricing model.

In Chapter 3, we consider the time- and state-dependent model of Helmes et al. (2013) where the capacity and the sales period are exogenously given values. Instead of a stock of items to be sold, we interpret the state of the system to be the fractional untapped market share. The objective is to identify a control that maximizes the net present value of the total revenue minus the accumulated advertising expenses. We use Pontryagin's maximum principle to derive the optimal open-loop control and also derive optimal feedback policies. To illustrate the general results we consider the *von Bertalanffy* growth function. We specify parameter constellations such that the optimal price control is either a *market skimming* or a *market penetration* policy. For the class of general *von Bertalanffy* models, we characterize the time point when prices peak.

The framework of Chapter 3 provides a tool to analyze and understand controlled adoption models. In particular, the controlled *von Bertalanffy* model gives insight into diffusion processes when the impact of the internal influence changes over time. This insight allows for a more general modeling of controlled adoption processes and thus helps improving the understanding of such processes. Due to the close relationship between new-product adoption models and inventory control problems, Chapter 3 makes a contribution to both research areas.

In Chapter 4, we extend the analysis of one-period models to multiple periods and long-term models. We introduce fixed order cost that arise once per cycle when the inventory is refilled at the beginning of a cycle. In Section 4.2, we assume that the profit maximizing (price-advertising) control we derived in Chapter 2 is applied during a cycle; in Section 4.3, we assume the revenue maximizing control from Chapter 3 to be applied. In both subsections, the cycle length and the capacity are considered to be decision variables. We distinguish between the maximization of the present value of N identical cycles and the maximization of the average profit per time unit. We derive conditions that guarantee the existence of a (unique) optimal pair of cycle length and capacity value and we refine these conditions for special parameter settings. Beside various examples that illustrate our results, we derive structural properties of the optimal results. For example, if the profit maximizing control of Chapter 2 is applied, we show that if the optimal advertising rate lies above a threshold value, then the maximized average profit in this market (where advertising is effective) is greater than the maximized average profit in a similar market where advertising has no effect; rephrased in everyday language: a monopolist benefits from advertising if the advertising level is sufficiently large. Moreover, the benefit goes along with a shorter average optimal cycle length (if advertising is effective), and, consequently, customers also benefit from advertising since (i) goods are fresh, and (ii) the (optimal) prices are not affected by advertising. In a particular parameter setting when the revenue maximizing policy is applied (Section 4.3), we derive explicit solution formulas for the optimal cycle length and the optimal capacity value. We refer to this result as the *endogenized Harris-Wilson* formula.

Our analysis in Chapter 4 assumes the profit maximizing and the revenue maximizing policies to be applied during a cycle. However, the fact that the policies are the profit maximizing ones is not essential for our analysis. Actually, any (feasible) price-advertising control that satisfies the optimality conditions characterizes a (sub)optimal order pair. Thus, our analysis applies to a broader class of inventory control systems than only those where the optimal control is applied. This observation leads directly to a possible extension of our models: a more general demand rate than the one of multiplicative form we consider throughout this thesis. In particular, the development of a model in which the demand rate underlies stochastic influences promises results that are of importance for management problems arising in practical applications. Other possible future research areas include numerical studies using real data, the generalization of our models to incorporate stochastic influences on inventory, the extension of the models to allow for backlogging and variable production costs.

We, however, concentrate our analysis on the isoelastic demand function that depends on the control variables price and advertising. Interpreting these controls in different ways it is possible to apply our models and results to a variety of different subjects. For example, if w is interpreted as *maintenance* control and if the price is fixed as in our model in Chapter 2, then the associated models in Chapter 4 can also be interpreted as a forestry rotation model. Another possible interpretation of our models is the one of a tool wear model. If p and w are particular controls that influence the process of wear and tear of tool or equipment, then the rate (1.1) can be interpreted as the change of this quality process x at time t, cf., for example, Sanjay et al. (2005). Interesting questions in this particular context are how to control this process in order to maximize a control-dependent objective function and when to renew the equipment and/or to which extent. Another (obvious) field of application for our models is the one of depleting exhaustible resources by a monopolist. By abstracting from the advertising control one can ask what is the (long-term) profit maximizing pricing and *depletion scheme*; in that particular application, the monopolist chooses a period of production and the amount of resources to be extracted.

This dissertation enhances the understanding of coordinated optimal revenue management and inventory management decision processes. The models and results presented help answer theoretical and practical management questions.

Appendix: Marketing Models

In Section 3.1, the two classical advertising models of Nerlove and Arrow (1962) and Vidale and Wolfe (1957) are briefly discussed. The following Section 1 deals with the optimal price and the optimal advertising control in the Nerlove-Arrow model. Section 2 presents the controlled Vidale-Wolfe model which was first considered by Sethi (1973), see also Sethi (1974). For a more detailed discussion of the models and derivation of the results we refer to the original literature. Moreover, Chapter 7 in Sethi and Thompson (2000) presents a comprehensive summary of both models, and applies control theory techniques to derive the results.

Unless otherwise stated the notation refers to Chapter 3. This includes the meaning and interpretation of parameters and variables. For general motivation and interpretation of the models we refer to Section 3.1. The subscript '$_{NA}$' refers to the Nerlove-Arrow model, and the subscript '$_{VW}$' to the model of Vidale and Wolfe.

1 The Nerlove-Arrow Advertising Model

Nerlove and Arrow (1962) consider a firm's investment in advertising to build up a stock of goodwill: one dollar spent on advertising increases the goodwill by one unit. The stock of goodwill, the state of the system, depreciates at a constant rate $q \geqslant 0$. Let $G(t) \geqslant 0$ denote the level of goodwill at time t. It is assumed that G satisfies the ODE, $t \geqslant 0$,

$$\dot{G}(t) = w(t) - qG(t), \qquad G(0) = G_0, \tag{1}$$

where $w(t) \geqslant 0$ quantifies the advertising spending at time t; $G_0 > 0$ is the initial level of goodwill of the firm. The demand λ for a product at time t depends on the price p charged at that time, the stock of goodwill $G(t)$, and an additional (time-dependent) variable $\mu(t)$ that can not be influenced by the firm, i.e.,

$$\lambda_{NA}(t) = \lambda(p(t), G(t), \mu(t)). \tag{2}$$

The total production cost c_{NA} is assumed to be a function of the demand, $c_{NA} =$

$C(\lambda_{NA})$, such that the net profit ν_{NA} - revenue minus production costs - is given by

$$\nu_{NA}(t) = p(t)\lambda_{NA}(t) - C_{NA}(\lambda_{NA}(t)). \tag{3}$$

Let $r \geqslant 0$ denote the constant (continuous) discount rate. The firm wants to maximize the present value of net profits over an infinite time horizon by choosing appropriate controls p and w, i.e., the objective of the firm is

$$\max_{p(t)\geqslant 0, w(t)\geqslant 0} \left\{ V_{NA} = \int_0^\infty e^{-rt} \left[\nu_{NA}(t) - w(t)\right] dt \right\}, \tag{4}$$

subject to equation (1). Since the net profit ν_{NA} is only affected by the current price, Nerlove and Arrow (1962) argue that the maximum can be found by first maximizing V_{NA} with respect to the price (assuming G to be fixed). Then, V_{NA} is maximized with respect to w when the price p is replaced by the optimal price p_{NA}. This approach is similar to the approach we take in Chapter 2.

The optimal price p_{NA} is characterized in terms of the price elasticity of demand $\varepsilon = -(p/\lambda_{NA})(\partial\lambda_{NA}/\partial p)$ and the marginal cost of production, i.e.,

$$p_{NA} = \frac{\varepsilon}{\varepsilon - 1} C'(\lambda_{NA}), \tag{5}$$

where C' denotes the derivative with respect to λ_{NA}. Equation (5) is the *usual* price formula for a monopolist, cf. (2.15).

Let $\delta_{NA} = (G/\lambda_{NA})(\partial\lambda_{NA}/\partial G)$ denote the elasticity of demand with respect to goodwill. Nerlove and Arrow show that (assuming technical conditions) the optimal stock of goodwill G^* (at time t) must satisfy

$$\frac{G^*}{p_{NA}\lambda_{NA}} = \frac{\delta_{NA}}{\varepsilon(r+q)}.$$

This equation is another example of a dynamic version of the Dorfman-Steiner relation, cf. (2.14). Note, the optimal price as well as the associated sales rate depend on $\mu(t)$. Thus, G^* generally also varies over time. If $\mu(t) = \mu$, the optimal goodwill level $G^* = G^*(\mu)$ is a constant. Depending on the initial value G_0, the optimal advertising policy w_{NA} is to reach G^* as fast as possible. If $G_0 < G^*$, this is achieved by an impulse control at $t = 0$. Once the optimal stock of goodwill has been attained, only the depreciation will be compensated, i.e., $w_{NA}(t) = qG^*$. If the initial level of goodwill exceeds the optimal one, then $w_{NA}(t) = 0$ is optimal until G^* is reached.

In the time-inhomogeneous case when μ does depend on t, if $G_0 < G^*(\mu(0))$, it is still optimal to apply an impulse control in order to jump from G_0 to $G^*(\mu(0))$ immediately. Also, if $G_0 > G^*(\mu(0))$, then $w_{NA}(t) = 0$ for $0 \leqslant t \leqslant \tau_{NA}$, where τ_{NA} is a solution of the equation $G_0 e^{-q\tau_{NA}} = G^*(\mu(\tau_{NA}))$. Once $G^*(\mu(t))$ has been attained, the optimal *replacement* policy is given by $w_{NA}(t) = \dot{G}^*(\mu(t)) + qG^*(\mu(t))$, $t \geqslant \tau_{NA}$, where this value is assumed to be nonnegative. Nerlove and Arrow point out, that "the optimal solution becomes complicated" if the nonnegativity property is not fulfilled.[1] They refer to Arrow et al. (1958) who studied a special case of the problem without depreciation in finite time. But even in this case the optimal solution (algorithm) can not be described in simple form.

2 The Vidale-Wolfe Advertising Model

Vidale and Wolfe (1957) do not make use of the idea of goodwill. Motivated by their empirical findings they describe the relationship of sales and advertising in terms of three parameters:

- The sales decay constant q_{VW}.
- The saturation level of the market $\mathfrak{M}_{VW}$.
- The response constant $\tilde{\mu}_{VW}$.

Vidale and Wolfe argue that the sales rate λ_{VW} reacts positively on advertising spending w via the response constant $\tilde{\mu}_{VW} > 0$.[2] On the other hand, there is a negative influence on the sales rate due to forgetting effects, product obsolescence, or competing advertising captured by the constant $q_{VW} \geqslant 0$. The saturation level $\mathfrak{M}_{VW}(t)$ denotes the maximum potential of the rate of sales at time $t \geqslant 0$. Therefore, $\mathfrak{M}_{VW}$ is also called the *market potential*. Based on experimental studies, Vidale and Wolfe (1957) characterize their model by the differential equation, $0 \leqslant t \leqslant T$, T finite,

$$\dot{\lambda}_{VW}(t) = \tilde{\mu}_{VW} w(t) \left(1 - \frac{\lambda_{VW}(t)}{\mathfrak{M}_{VW}}\right) - q_{VW}\lambda_{VW}(t), \tag{6}$$

where w is the advertising rate and λ_{VW} is the sales rate, both (observed) at time t. The original motivation was to understand market behavior and to obtain reliable estimates of the parameters of interest, $q_{VW}, \mathfrak{M}_{VW}$, and $\tilde{\mu}_{VW}$.

[1] Nerlove and Arrow (1962), p. 138.
[2] The interpretation of this response constant is the number of sales per advertising dollar spent.

Sethi (1973) was one of the first to consider a controlled Vidale-Wolfe model, where the goal is to determine an optimal advertising control that maximizes a particular objective function. He reformulates the problem in terms of *market shares* $x_{VW} := \frac{\lambda_{VW}}{\mathfrak{M}_{VW}}$. Defining $\mu_{VW} := \frac{\tilde{\mu}_{VW}}{\mathfrak{M}_{VW}}$ equation (6) can be rewritten as

$$\dot{x}_{VW}(t) = \mu_{VW} w(t)\big(1 - x_{VW}(t)\big) - q_{VW} x_{VW}(t), \quad x(0) = x_0, \tag{7}$$

where $x_0 \geqslant 0$ denotes the initial market share. Equation (7) constitutes the state equation of the control system. Note, the dynamic (7) differs from the state equation we consider in Chapter 3 since $\dot{x}_{VW}(t)$ denotes the *change* in the (relative) market share measured as a portion of the overall market sales rate, i.e., changes of the proportional sales rate. The state equation (3.5), however, characterizes the change of the untapped market share, which will only take nonpositive values.

Sethi does not consider a particular price control but rather assumes a maximum sales revenue $p_{VW} > 0$ associated with $x_{VW} = 1$. Then, the product $p_{VW} \times x_{VW}$ denotes the revenue function for $x_{VW} \in [0, 1]$. Let $\bar{w}_{VW} > 0$ be the maximum allowable advertising rate at any time; $\bar{w}_{VW}$ may be finite or infinite. Moreover, let $x_T \in [0, 1]$ denote the target market share at the end of the planning horizon; $r > 0$ denotes the (continuous) discount rate. Hence, the optimal control problem is

$$\max_{0 \leqslant w(t) \leqslant \bar{w}_{VW}} \left\{ V_{VW} = \int_0^T e^{-rt} \left(p_{VW} x_{VW}(t) - w(t) \right) dt \right\}, \tag{8}$$

subject to (7) and the terminal state constraint $x_{VW}(T) = x_T$. Note, since $x_0 \in [0, 1]$ the market share automatically satisfies $0 \leqslant x_{VW}(t) \leqslant 1$ for all $t \in [0, T]$.

Sethi (1973) derives the optimal control w^*_{VW} for this fixed end point problem, see also Sethi and Thompson (2000). In particular, he shows that there exists a constant optimal market share level $x^s_{VW} \in [0, 1)$; x^s_{VW} is a function of the parameter values q_{VW}, μ_{VW}, p_{VW}, and r. Similar to the optimal control in the Nerlove-Arrow model, it is optimal to attain this optimal level x^s_{VW} as fast as possible and to keep this level as long as possible. Sethi characterizes the optimal control depending on the time horizon T, the initial market share x_0, and the maximum allowable advertising level $\bar{w}_{VW}$.

For example, if $x_0 < x^s_{VW}$ and $\bar{w}_{VW}$ is infinite, then an impulse control is optimal to bring the state to its optimal level immediately. If the value $\bar{w}_{VW}$ is finite, then $w^*_{VW}(t) = \bar{w}_{VW}$ is optimal until the level x^s_{VW} has been attained. This assumes that the time horizon T is large enough to reach the optimal level *and* to satisfy the terminal state control, respectively, that the maximum allowable advertising level is large enough.

If x^s_{VW} has been reached, it is optimal only to replace the loss due to the decay q_{VW}. Since at the optimal market share level $\dot{x}_{VW}(t) = 0$ must hold, equation (7) can be rearranged to obtain the optimal replacement policy.

In order to satisfy the terminal state condition, one has to deviate from the market share level x^s_{VW}.[3] For example, if $x_T < x^s_{VW}$, it is optimal to stop advertising early enough ($w^*_{VW}(t) = 0$) such that the market share hits the target value at time T. When the time horizon is not large enough or if the allowable advertising level is too small so that it is not possible to attain the level x^s_{VW} *and* satisfy the terminal state control $x_{VW}(T) = x_T < x^s_{VW}$, Sethi shows a bang-bang control to be optimal.

Moreover, Sethi (1973) also considers the problem when the end point is free. He derives the optimal control by use of the maximum principle. Sethi and Thompson (2000) also analyze the infinite horizon problem.

[3]Naturally, the target value and the value of the desirable market share may coincide, i.e., $x_T = x^s_{VW}$. Then, it is optimal to follow the replacement policy until time T.

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